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EDITORIAL

Welcome to the summer edition of SYMmetry*plus*! I hope that all readers who are engaged with mathematics examinations at this time of year manage to find some time to read this edition and that your results are what you hope for, or better!

I have now taken on the role of President of The Mathematical Association since our Easter conference at Loughborough University and it was a delight to meet and talk with many members, some of whom subscribe to SYMmetry*plus* through their association membership, or through SYMS. If you have not yet joined SYMS, make sure you read about it on page 15 and yes, it is far cheaper to join SYMS and get two journals that way for less than the cost of one!

I showed two Mexican friends around the Oxford sundials recently and we were all very impressed by the Pelican sundial in Corpus Christi College (see opposite). This magnificent piece of mathematical art was made by Charles Turnbull between 1579 and 1581. The date of 1605 probably refers to when he restored it. As well as a multitude of dials it contains details about finding the date of various religious ceremonies, such as Easter.

In this issue Arnold C Mann and Origami Pete have written a joint article about escaping after being marooned; Oliver Wilshaw describes what he found when investigating clocks and primes at his primary school; Paul Harris gives us another crossnumber; Jenny Ramsden and Graham Hoare tell us about Clairaut; Paul Stephenson of the Magic Mathworks Travelling Circus guides us through groups and Latin squares and Nigel Bufton explains how divisibility tests work. Fun for all!

Peter Ransom

THE PELICAN SUNDIAL

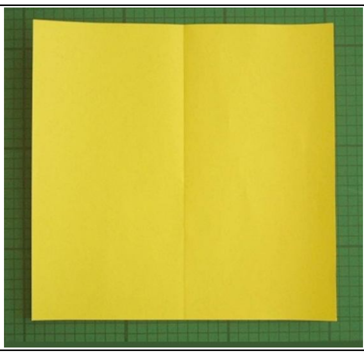
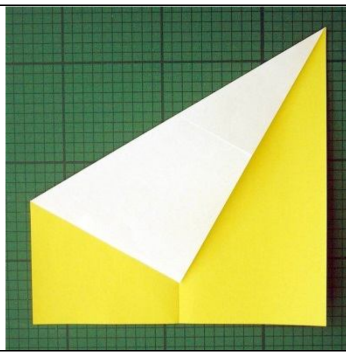
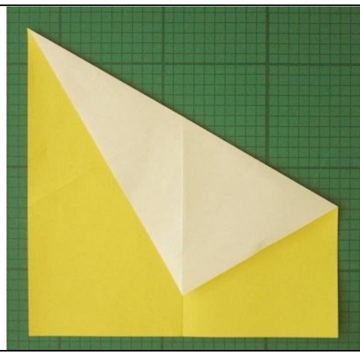
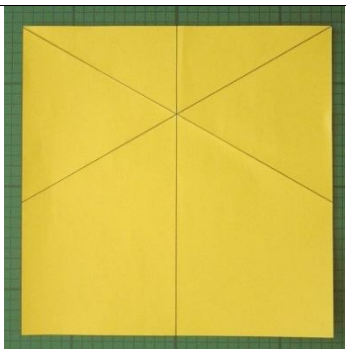
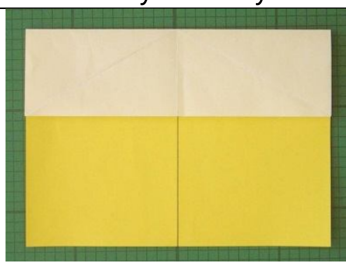
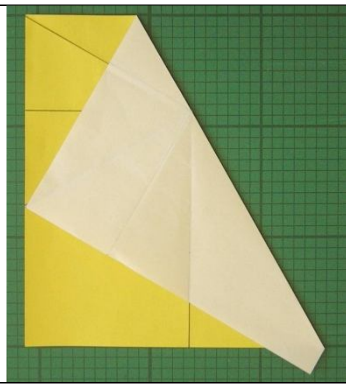
ESCAPE TO GUYANA

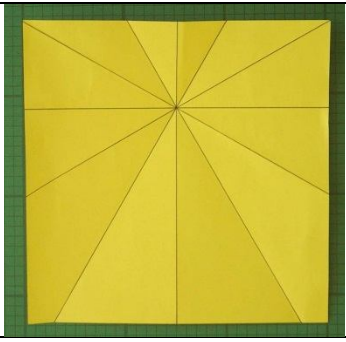
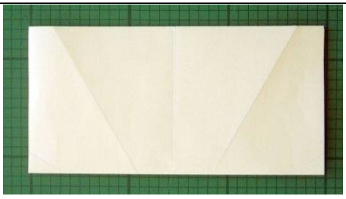
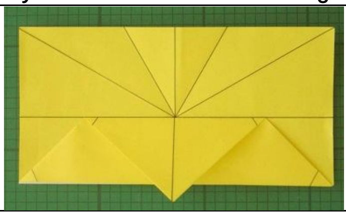
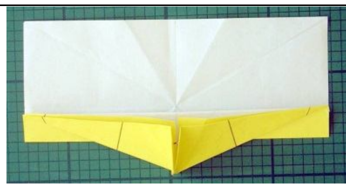
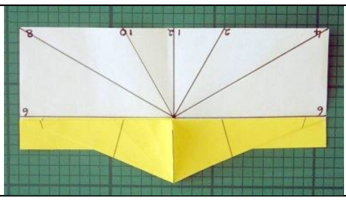
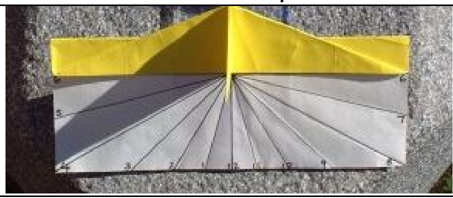
… continued from issue 50

Life after being marooned on an island off the coast of Guyana on the north coast of South America was hard. Black Bart, captain of the Royal Rover, had abandoned us for gambling on his ship in contravention of the contracts we had signed. We knew that if we had any chance of getting off this living hell then we would have to navigate across part of the Atlantic Ocean to get to the mainland. Unfortunately we had been left no navigational instruments to help. Most of our time was spent keeping ourselves alive, seeking food from local plants and animals. There was a fresh water supply, which meant that we did not have to catch rainwater, so things could have been worse. Of course we had no timepieces as common pirates such as me had no need of such trinkets but we knew that we needed some indication of the time if we were to ascertain our location and navigate to Georgetown, the capital of Guyana.

Fortunately Origami Pete had been marooned with us and he knew a thing or two about folding paper (he had managed to smuggle some paper off the ship when we were landed) and managed to find a way of folding a simple sundial which gave us the means of telling the time in the day. This was important as we then had something which we could orientate at local noon (when the sun was at its zenith) and that would allow us to determine true north and hence the other points of the compass.

Investigating the local trees we noticed that we could smell rosewood and it was not hard to track down the trees that smelt most. Rosewood oil comes from the tree *Aniba rosaeodora* which was also used by the indigenous population of the Amazon basin to make canoes! So we had the raw materials to escape and set Origami Pete to work. He felt it was important for everyone to know how to make a sundial, so we all had to follow the following instructions.

		
1 Fold a square sheet in half.	2 Fold through one corner so the other corner meets the crease. This means the angle at the top is exactly 60°. Why?	3 Unfold and repeat with the opposite corner.
		
4 Open it out and mark the creases. They will form six 60° angles.	5 Fold the top over through the point where the lines intersect.	6 Bisect each 60° angle by folding a crease onto its neighbour.

		
7 Open it out and mark the creases. They will form twelve 30° angles.	8 Fold the sheet in half.	9 Fold the top two corners to the middle of the bottom fold.
		
10 Fold down the top of the triangle, creasing on the horizontal line	11 Bring the top of the triangle up and press in at the sides so that you get a small triangle perpendicular to the horizontal plane.	12 Turn it over and fold back so that the bottom edge in 11 lies on the other side of the horizontal line mentioned in 10.
		To set the sundial correctly, wait until noon. This will be when the shadow of a stick is shortest. When that occurs, move the sundial so the shadow falls on the 12 line. The direction of the 12 line will then be in a north-south direction with north at 12.
13 Draw and mark the hour lines as shown	14 Incline the sundial so that the edge that casts the shadow on the time lines is inclined at an angle equal to your latitude (use a satnav or visit http://www.satsig.net/maps/lat-long-finder.htm.	

Therefore our mathematics was vital to our well-being. Without our knowledge of equilateral triangles and being able to bisect angles by folding we would not have been able to construct the hour lines on our sundial, necessary because the sun appears to travel through 360° in 24 hours, or 30° in 2 hours. This, together with a bit of Earth/Sun knowledge meant we could find true north. So we now had our time machine which we could also use as a compass. We now just needed something to measure our latitude and a boat. It took us some time to carve out a canoe from the wood and make our navigation instrument, which will feature in a future article.

To be continued ...

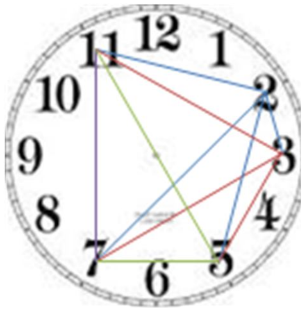
Arnold C. Mann and Origami Pete

Quickie 6

Using only addition, subtraction, multiplication, division and possibly brackets,
make 24 from the numbers 3, 3, 8 and 8.

CLOCK FACE PRIMES

If you imagine a clock face and joined up the primes, you would use 10 lines.



The pairing should be:

Starting with 2: 2-3, 2-5, 2-7, 2-11

Starting with 3: 3-5, 3-7, 3-11

Starting with 5: 5-7, 5-11

Starting with 7: 7-11

However, if you used a clock with, for example, numbers up to 30, you need a formula to work out the amount of lines required to join the prime numbers.

Let n represent the number of primes in the range of numbers you are using, and let l represent the number of lines. Then $l = \frac{n(n-1)}{2}$. Why this? It's because of the triangle number formula.

Imagine the primes as $p_1, p_2, p_3, \dots, p_{n-1}, p_n$ if n is the number of primes. $p_1$can connect to a total of $n-1$ other primes, as it is one itself. p_2 can then connect to $n-2$ other primes, because it is already connected to p_1, and, of course, it is one itself. If you continue, you find that p_x can connect to $n-x$ other primes. If you find the triangle of $n-1$, then you have $(n-1)+(n-2)+\cdots+1$. This is exactly the format of the above; p_1 (2) has $n-1$ primes it joins to (4). p_2 (3) joins to $n-2$ primes (3).

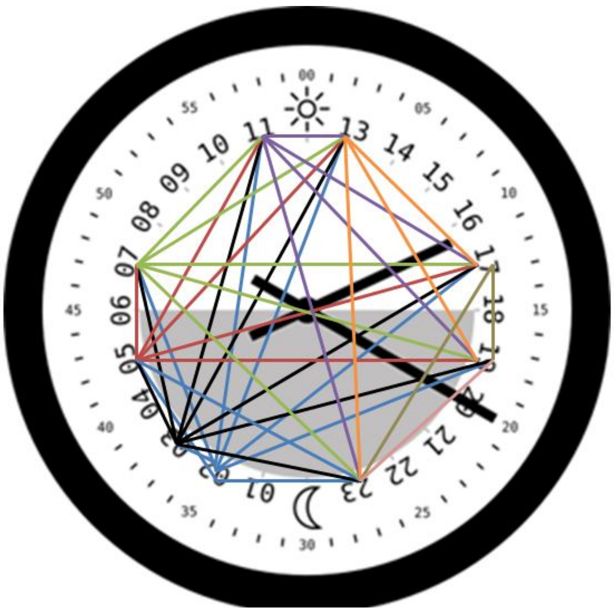
As the triangle number formula is $T_n = \frac{n(n+1)}{2}$, $T_{(n-1)} = \frac{(n-1)((n-1)+1)}{2} = \frac{n(n-1)}{2}$. This leaves us with the fact that l equals $\frac{n(n-1)}{2}$.

Let's try it out on a few strange clocks.

Imagine a clock that has the numbers 1-24.

It has 9 primes: 2, 3, 5, 7, 11, 13, 17, 19 and 23; so $p = 9$.

Therefore $l = \frac{p(p-1)}{2} = \frac{9(9-1)}{2} = \frac{9\times8}{2} = \frac{72}{2} = 36$.



Let's prove it:

That looks like 36 to me!

This next one you'll have to imagine; you can't find an image of an 86-hour clock! The primes are listed on the next page.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, and 83 are all the primes; there are twenty-three, so the formula is

$l = \frac{p(p-1)}{2} = \frac{23(23-1)}{2} = \frac{23\times 22}{2} = \frac{506}{2} = 253$. That is a lot of connections!

Goodbye!

Bibliography:

24-Hour '12'-at-the-top Analog Clock. (n.d.). retrieved January 23, 2013 from StatisFree Info.: http://staticfree.info/projects/24h_clock/

Oliver Wilshaw

SUDOKU

2				6	
			5		2
	3				1
6				5	
			2	1	6

In this Sudoku, each row, column and block of colour contains the digits 1 to 6

Peter Ransom

WHEN MORE COST LESS

Sometimes buying more than you need works out cheaper than buying exactly what you need!

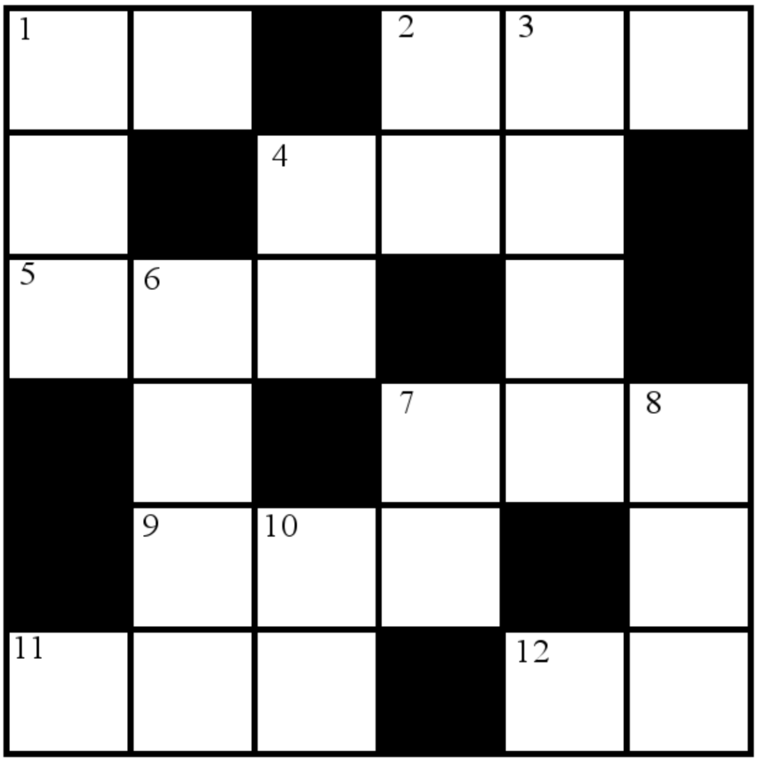
I was painting the kitchen and utility room and I estimated that to paint two coats of paint I would need paint to cover over $40m^2$. So I went to a well-known DIY store and looked at what was on offer. On the 2.5 litre tin of paint it said it would cover $40m^2$, so I did not think there would be sufficient paint in one tin. There was an offer on of 3 tins for the cost of 2, so I took 3 tins, thinking I would need at least 2 and that a spare would not come amiss. This cost me a total of £47.96, each tin costing £23.98

Anyway, I managed with just one tin, so took the others back for a refund, expecting just £23.98 as a refund as I had only used one tin. However they refunded me £31.98 because they gave me back ⅔ the cost of each tin! So the single tin I used cost me only £15.98 instead of £23.98. I tried to explain this, but was told the computer said I had received the correct refund, so not to worry! What fraction of each tin should they have refunded?

Quickie 7

Write down the next two terms in the sequence 1, 11, 21, 1211, 111221, ...

CROSS NUMBER PUZZLE



	Across		Down
1	A perfect square and also a perfect cube. (2)	1	A Fibonacci number plus 1. (3)
2	A power of 2. (3)	2	A number whose second digit is double its first. (2)
4	A number whose digits are all the same. (3)	3	$\sum_{j=0}^{n} 4^j$ for some value of *n*. (4)
5	A Fibonacci number which is a perfect square. (3)	4	5 across minus 100. (2)
7	A number whose digits add up to 10. (3)	6	The sum of the cubes of three consecutive integers. (4)
9	The sum of the first *n* integers for some value of *n*. (3)	7	A number which is the square of the sum of its digits. (2)
11	The largest 3-digit prime number. (3)	8	A perfect square whose middle digit is the sum of the other two digits. (3)
12	The square-root of 8 down. (2)	10	A number which is 3 less than 10 times its second digit. (2)

Paul Harris

Quickie 8

If $\sqrt{1+\sqrt{1+\sqrt{1+n}}} = 2$ calculate n

ALEXIS CLAUDE DE CLAIRAUT

In 2013 we are celebrating the three hundredth anniversary of the birth of the mathematician Alexis de Clairaut (known nowadays as "Clairaut"). Clairaut was born on 7 May 1713 in Paris and was a youthful prodigy, one of a family of 20 children but the only one who survived to adulthood. He and a younger brother were educated at home and both of them were very capable at mathematics, taught to them by their father, Jean-Baptiste, who was a mathematics lecturer and a member of the Berlin Academy. By the age of ten, Clairaut had already read L'Hôpital's works on the infinitesimal calculus and on conic sections, works that most adult students would normally tackle. At only 12 Clairaut wrote a paper on four geometrical curves and read an account of these curves, *Quatre problèmes sur les nouvelles courbes,* to the Parisian Academy of Sciences in 1726. His younger brother also read a paper to the Academy in 1730 when he was just 14 years old, but sadly he died at age 16 of smallpox.

When Clairaut finished his paper, *Recherches sur les courbes a double courbure*, he was only 16. It was published in 1731 and this paper secured his admission to the Academy when he was legally still too young to join, being just 18 years of age. In the same year he proved a theorem stated by Newton (that every cubic is a projection of one of five divergent parabolas).

While at the Academy, Clairaut became interested in mathematics of the world's measurements, in particular the works of Cassini and Newton in measuring the meridian, or circumference of the world. Cassini and Newton held different views - Newton (and Huygens) believed from theory that the world was an oblate spheroid (flattened at the poles), whereas the Cassinis (father and son) believed the world was elongated at the poles, having measured the distance from Dunkirk to Perpignan in 1712. In order to ascertain correct measurements to decide the case once and for all, two expeditions were sent out, one to Lapland and one to Peru, to collect information that could be used to measure the length of one degree of latitude along a line of longitude (a "meridian arc"). This information would then be used to determine the shape of the earth by measuring its curvature - once near to the North Pole and once near to the equator. The Peruvian expedition was led by Charles Marie de la Condamine, but Clairaut accompanied Pierre Louis Maupertius on the Lapland expedition, setting out from Paris on 20 April 1736 along with other scientists including Le Monnier, Camus, the Abbé Outhier and Celsius. They travelled to take their measurements in the Tornionlaakso Valley, which lies close to the border of Sweden and Finland. The expedition returned on 20 August 1737 with evidence to show that Newton's theory was correct; the world was indeed flattened at the poles. This was a great victory for those supporting Newton's theory and it sparked many studies of hydrodynamics by Clairaut, Maclaurin, Euler and others. Following the success of the expedition, Clairaut was elected a Fellow of the Royal Society of London in November 1737.

Clairaut turned his attentions after this to celestial mechanics, being interested in the work of Newton in particular. He contributed fundamentally to the dissemination of Newton's ideas about gravity and light on the Continent; he helped the Marquise du Châtelet translate Newton's *Principia* into French and in preparing the explanations that accompanied the text. This project was started in 1745 and continued until part of it was published in 1756. The translation was very faithful to the original and in its second volume there is an explanation of the system of the world that clearly owes much to Clairaut's own publications.

Clairaut had a multitude of results published by different Academies. His monographs and text books published were *Recheres sur les courbes à double courbure* (1731, Paris); which were dealt with by methods of infinitesimals (curves with a double curvature), which ran to many editions and languages. *Elémens de géométrie* (Paris 1741) and *Elémens d'algèbre* (Paris 1746) were the most popular, being intended to be part of a plan to improve the teaching of mathematics. So popular was the algebra book that it had run to six editions by 1801 and was used in French schools for many years.

Differential equations named after Clairaut first appeared in 1734 in a publication by the Parisian Academy. In 1743, *Théorie de la figure de la terre* was published as a result of the Lapland trip, based on the results of Maclaurin on homogenous ellipsoids. Maclaurin had shown that a mass of homogeneous fluid set in rotation about a line through its centre of mass would, under the mutual attraction of its particles, take the form of a spheroid. Clairaut's work looked at these spheroids and contained the proof of his formula for the accelerating effect of gravity in a place of latitude *l,* namely $g = G\left(1 + \frac{5}{2}m - \varepsilon\right)\sin^2 l$, where *G* is the value of equatorial gravity, *m* is the ratio of the centrifugal force to gravity at the equator and ε is the ellipticity of a meridian section of the earth. This remarkable theorem is now named after Clairaut. It allows the ellipticity of the earth to be calculated from surface measurements of gravity.

In 1752 he published *Théorie de la lune*, in which he applied modern analysis to lunar motion and to three body motion in general. For this publication he won a prize from the St Petersburg Academy. This publication contained the explanation of the motion of the lunar apsis, a problem which had worried scientists and astronomers greatly for some time. Newton himself had stated that his theory of gravity failed for the moon and several mathematicians tried to calculate the error. Clairaut had initially thought the only way to correct the error was to add an inverse fourth power. However, having checked his calculations, Clairaut realised that if he extended his approximations to the third order, rather than stopping at the second, he would have results that matched the observations.

Clairaut applied his knowledge of the three body problem to the orbit of Halley's comet and thus was able to predict when the comet would next come closest to the sun. He stated this would happen about 15th April 1759, which turned out to be just one month after it actually happened on 13th March. When we consider the lack of mechanical calculating devices for assisting with such calculations at this time, being only one month in error in such a calculation is impressive. Indeed, Clairaut was congratulated widely for his work and there was even a suggestion that Halley's comet should be renamed Clairaut's comet.

Clairaut also applied the process of differentiation to the differential equation now known by his name and detected its singular solution. The equation was $y = px + f(p)$, where $p = \frac{dy}{dx}$.

Alas, despite his many successes in the academic world, Clairaut was drawn increasingly to the social whirl of Paris, which hindered his scientific work as he grew older. Sadly he died after a short illness in Paris on 17 May 1765, aged just 52 years.

Jenny Ramsden

Quickie 9

An ant starts at (0, 0) and can only move one unit up or one to the right at a time. For $n \geq 0$, how many ways are there for the ant to get to (2, n)?

CLAIRAUT'S EQUATION

Clairaut's equation is given by the formula $y = xp + f(p)$, where $p = \dfrac{dy}{dx}$.

To solve this we differentiate the equation with respect to x.

Thus $p = p + x\dfrac{dp}{dx} + \dfrac{df(p)}{dp}.\dfrac{dp}{dx}$ (Note the use of the chain rule).

So, either we have that $\dfrac{dp}{dx} = 0$ or $x + \dfrac{df(p)}{dp} = 0$.

From the first of these, $p = c$ (where c is a constant) and so $y = cx + f(c)$ (by substituting this solution in the original equation). The solution we have derived represents a family of straight lines as c varies.

The other solution, given parametrically, is $x = -\dfrac{df(p)}{dp}$, $y = -p\dfrac{df(p)}{dp} + f(p)$ (by substituting for x in the original equation). We can easily show that this <u>singular</u> solution, as it is called, is the envelope of the family of straight lines given by the first solution. Using the equation $y - y_1 = m(x - x_1)$ for a straight line passing through the point (x_1, y_1) with gradient m, the equation of the tangent line at a point for which $p = c$ is given by

$$y - \left(-cf'(c) + f(c)\right) = c\left(x - \{-f'(c)\}\right).$$

Note that the gradient of the tangent line is p (= c) and $f'(c)$ means $\dfrac{df(p)}{dp}$ when $p = c$.

Thus $y = cx + f(c)$, as required.

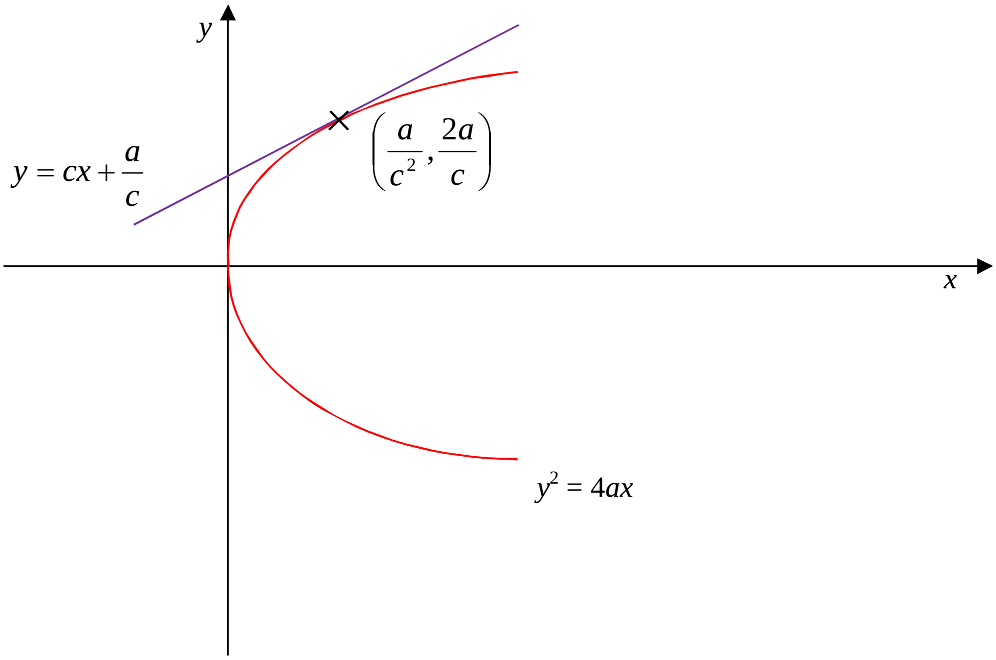
Example

Consider a Clairaut equation $y = px + \dfrac{a}{p}$, where a is a constant. Call this equation (1).

As in the general case, differentiating with respect to x yields $\dfrac{dp}{dx}\left(x - \dfrac{a}{p^2}\right) = 0$. So $\dfrac{dp}{dx} = 0$ or $x = \dfrac{a}{p^2}$.

From the first of these, $p = c$, so $y = cx + \dfrac{a}{c}$ (from equation (1)). From the second of these and (1) we have $y = \dfrac{2a}{p}$. So the singular solution is given by $x = \dfrac{a}{p^2}$, $y = \dfrac{2a}{p}$. Eliminating the parameter p we arrive at the familiar Cartesian equation, $y^2 = 4ax$, of the standard parabola.

It is straightforward to verify that the line whose equation is $y = cx + \dfrac{a}{c}$ is tangent to the parabola at the point $x = \dfrac{a}{c^2}$, $y = \dfrac{2a}{c}$. Thus the family of lines $y = cx + \dfrac{a}{c}$ envelopes the parabola $y^2 = 4ax$.



Graham Hoare

FROM LATIN TO GRAECO-LATIN part 1

Introduction

In a *Latin square of order k* each row and column contains all *k* elements.

To put that in the negative, no element occurs twice in a row or column. The issue of *SYMmetry*plus for summer 1993 discusses a famous example. And in the 20 years which have passed, a game based on Latin squares, Sudoku has achieved unique popularity. In case you think there isn't much maths in Sudoku, go to http://nrich.maths.org, enter 'Sudoku' in the search box and you will find 43 different activities based on it. In case you're still not convinced, get a copy of this recently published book of 213 pages: 'Taking Sudoku Seriously: The math behind the world's most popular pencil puzzle' by Jason Rosenhouse and Laura Taalman.

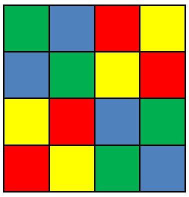
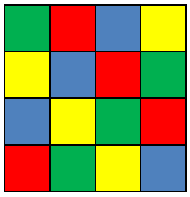
A *Graeco*-Latin square is a pair of Latin squares in which each pairing of corresponding elements is distinct. Thus in an order *k* Graeco-Latin square all k^2 possible pairings occur. Leonhard Euler was the first to study Graeco-Latin squares seriously. The name comes from his practice of using 'a, b, c, ...' for one set of elements and 'α, β, γ, ...' for the other.

Finding these so-called *orthogonal pairs* makes a challenging puzzle when $k = 4$, (the first composite number). My starting-point for 'Graeco-Latin Squares' (*SYMmetry*plus issue 17, spring 2002) was the challenge 'Aunty's Teacups' in the NRICH hands-on maths roadshow, which I had adopted in my own. I have therefore had the chance over several years to observe how students tackle the problem.

It has so many interesting aspects that I want to spend the next 7 sections 'unpacking' my original piece by going back to the Latin squares themselves (sections **1** to **4**) before assembling the orthogonal pairs (sections **5** to **7**). [Editor's note: sections **3** to **7** will appear in later issues.]

From Latin to Graeco-Latin 1: Counting the different-*looking* Latin Squares of Order 4

Here are 2 different-looking squares:

How many are there altogether?

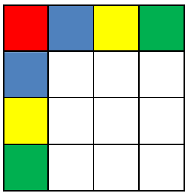
Imagine we are building the square from the top left corner. We start with the top row.

We are free to choose the first cell in 4 ways, the second in 3, the third in 2, and the fourth in 1. Then we work down the first column in the same way:

4	3	2	1
3			
2			
1			

Here is the number of choices for each cell.
To find the number of possible arrays of this shape we just form the product (4 x 3 x 2 x 1) x (3 x 2 x 1) or 4! X 3! = 144. (We read 4! as "four factorial" or, more dramatically, "four shriek".)

We fix one such array and find how many different ways we can arrange elements in the 3 x 3 square left:



Experiment with colours on squared paper.
You should find there are just 4 ways to fill the rest of the grid.

If there are 144 different 'L'-frames and for each there are 4 ways to complete the square, there must be 4 x 144 = 576 different Latin squares of order 4.

Apply this method to Latin squares of order ***3***. How many are there of those?

From Latin to Graeco-Latin 2: Defining Latin Squares

So, there are 576 different-*looking* Latin squares of order 4.

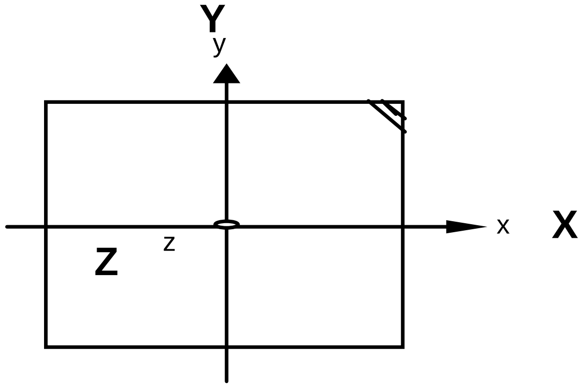
If I point to 2 of these and say, "Ah, but these are really the same", what do I mean?

I mean that, if they were tables displaying information, the information they displayed would be the same. But that too requires explanation:

Consider a particularly important Latin square of order 4. It records what happens when you take a rectangle of card and perform symmetry operations on it in sequence. (Those who read Martin Perkins' piece 'The Algebra of the Equilateral Triangle' in the issue of *SYMmetry*plus for summer 1994 will be familiar with the ideas here.)

We take our card - slightly dog-eared so that we can keep track of what happens to it.

We align its symmetry elements with coordinate axes - 2 mirror lines (which we can think of as axes of half-turn symmetry lying in the plane of the card) with the x- and y-axes and an axis of half-turn symmetry with the z-axis:



<u>Key to symmetry operations</u>

I	do nothing
X	flip (rotate a half-turn) about the x-axis
Y	flip (rotate a half-turn) about the y-axis
Z	rotate a half-turn about the z-axis

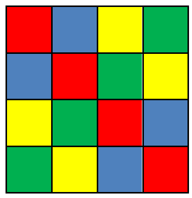
We find that when we perform one such operation after the other, the result is simply one of the single operations in our set. We call such a special set a *group*. Our set is the *symmetry group of the rectangle* but the same pattern might relate many other sets of things: because we are *abstracting* the pattern from any particular context, we call it an *abstract group*. This one is called the *Klein four-group* after the mathematician Felix Klein.

It is defined by the table which records the result of combining (or *composing*) operations. (By convention we perform the operation along the top followed by the one down the side.) Check its correctness with a playing card:

	I	X	Y	Z
I	I	X	Y	Z
X	X	I	Z	Y
Y	Y	Z	I	X
Z	Z	Y	X	I

Notice that the table is a Latin square.
We can swap colours for the letter labels:

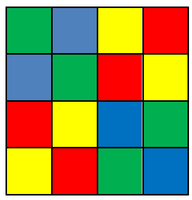
A



Now - and here is the point of this example - the way we arrange the row and column headings is purely a matter of convention. Let's permute them and observe the result:

	I	Y	X	Z
Z	Z	X	Y	I
X	X	Z	I	Y
I	I	Y	X	Z
Y	Y	I	Z	X

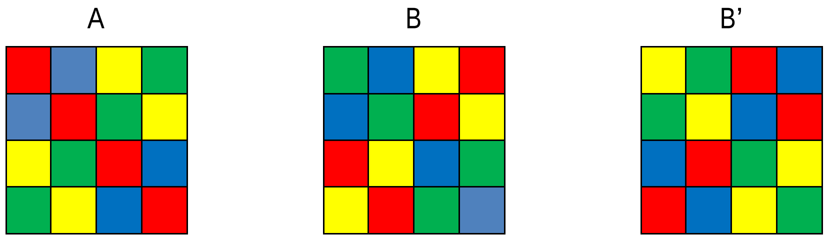
B



By our definition, A and B are the same.

Say we change our colour code:

<u>Key to symmetry operations</u>		B coding	B' coding
I	do nothing		
X	flip (rotate a half-turn) about the x-axis		
Y	flip (rotate a half-turn) about the y-axis		
Z	rotate a half-turn about the z-axis		



It makes no difference: A, B and B' are all the same square.

If we have two mathematical objects and the features of one exactly parallel those of the other, we call them *isomorphic.* A, B and B' are more closely related than that because one object is simply one of the others relabelled. We call them *automorphic*.

To be continued ...

Paul Stephenson

ANSWERS TO THE QUICKIES IN ISSUE 50

Quickie 1

Q: A clever stall keeper sells envelopes. Each standard pack contains 100 envelopes. It takes the stall keeper one second to take out an envelope. How much time will it take the stall keeper to count 75 envelopes?
A: 25 seconds since the clever shopkeeper removes 25 envelopes from a pack of 100.

Quickie 2

Q: Expand fully $(x^4 + 16)(x + 2)(x^8 + 256)(x^2 + 4)(x - 2)$
A: $(x^2 - 4)(x^2 + 4)(x^4 + 16)(x^8 + 256)$ $= (x^4 - 16)(x^4 + 16)(x^8 + 256)$
$= (x^8 - 256)(x^8 + 256)$
$= x^{16} - 65536$

Quickie 3

Q: $(1\tfrac{1}{2})(2\tfrac{2}{3})(3\tfrac{3}{4})(4\tfrac{4}{5})\ldots(19\tfrac{19}{20})(20\tfrac{20}{21})$ can be written as $p \times 19!$ What is the value of p?
A: When cancelled diagonally,
$(\tfrac{3}{2})(\tfrac{8}{3})(\tfrac{15}{4})(\tfrac{24}{5})(\tfrac{35}{6})\ldots.(\tfrac{360}{19})(\tfrac{399}{20})(\tfrac{440}{21})$ becomes $(\tfrac{1}{2})(\tfrac{2}{1})(\tfrac{3}{1})(\tfrac{4}{1})\ldots.(\tfrac{18}{1})(\tfrac{19}{1})(\tfrac{440}{1})$
which is 220 × 19! $\Rightarrow p = 220$

Quickie 4

Q: The co-ordinates of A, B and C are $(5, 5)$, $(2, 1)$ and $(0, k)$.
Find the value of k which makes $AC + BC$ as small as possible.
A: To make $AC + BC$ as small as possible, reflect B in the y-axis to give point D.
C will be the intersection of the line AD and the y-axis. Hence (with a bit of work) $k = 2\tfrac{1}{7}$

Quickie 5

Q: a and b are positive integers such that $a^9b^5 - a^5b^9 = 505440$. What is the value of a^2b?
A: $a^9b^5 - a^5b^9 = a^5b^5(a^4 - b^4) = a^5b^5(a^2 + b^2)(a + b)(a - b)$
$505440 = 2^5 \times 3^5 \times 5 \times 13 = 3^5 \times 2^5 \times (3^2 + 2^2)(3 + 2)(3 - 2) \Rightarrow a = 3, b = 2 \quad \Rightarrow a^2b = \mathbf{18}$

DIVISIBILITY

It is likely that you know and use certain tests of divisibility. For example, you might well use the test involving the addition of the digits of a number and determine if this total is a multiple of 9. If it is, then we then know that the original number is divisible by 9. But why does this work?

Every number is made up of a sum of multiples of the powers of 10. The position of each digit in the number identifies which power of 10 we are to multiply by the digit. So in the case of 876 543, we know this represents:

$8 \times 10^5 + 7 \times 10^4 + 6 \times 10^3 + 5 \times 10^2 + 4 \times 10^1 + 3 \times 10^0$

Expanding the powers we have the following 6 terms in the representation:

8 x 100 000 + 7 x 10 000 + 6 x 1 000 + 5 x 100 + 4 x 10 + 3 x 1

Imagine we are to divide 876 543 by 9. We can do this by dividing each term in the expansion of the number and then add the results together. This may seem a rather inefficient way of dividing but let's look at how we might do this and what information it gives us.

The first element in our number is 8 x 100 000. We can divide the 100 000 by 9 and multiply the result by the 8.

100 000 divided by 9 is 11 111 remainder 1.

It means that 8 x 100 000 or 800 000 divided by 9 is 8 x (11 111 remainder 1).

This tells us that 800 000 divided by 9 is 88 888 remainder 8.

If we repeat this for each term and by now you will have seen a pattern:

8 x 100 000 + 7 x 10 000 + 6 x 1 000 + 5 x 100 + 4 x 10 + 3 x 1 divided by 9

88 888 rem 8 + 7 777 rem 7 + 666 rem 6 + 55 rem 5 + 4 rem 4 + 0 rem 3.

And now our answer is:

88 888 + 7 777 + 666 + 55 + 4 + 0 remainder (8 + 7 + 6 + 5 + 4 + 3)

97390 remainder 33

Of course when dividing by 9 we would never expect to get a remainder of 33. But look at how this remainder was derived. It is the sum of the digits in the number we were dividing. Also as this remainder is not a multiple of 9 it means the original number cannot be a multiple of 9. To arrive at the final answer we need to divide our remainder 33 by 9. This gives us the alternative 3 remainder 6 and we use this to arrive at our final answer:

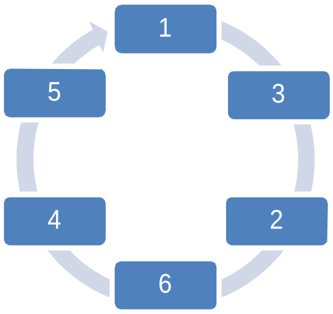
876 543 divided by 9 is 97393 remainder 6.

The process of dividing the powers of 10 by the divisor is one that can help us to find other tests of divisibility. One test you may not know is that for divisibility by 7.

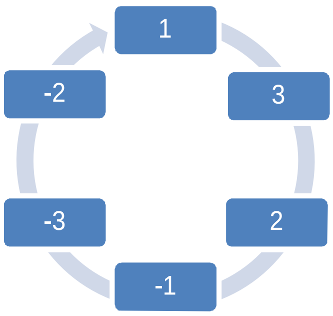
Let's divide the powers of 10 by 7 and see if we can identify a pattern. We shall start with 1, the unit and work our way up.

Power of 10	Quotient	Remainder
1	0	1
10	1	3
100	14	2
1 000	142	6
10 000	1428	4
100 000	14285	5
1 000 000	142857	1
10 000 000	1428571	3
100 000 000	14285714	2
1 000 000 000	142857142	6
10 000 000 000	1428571428	4
100 000 000 000	14285714285	5

At this point we see a recurring pattern in the quotient, the result of the division, and in the remainders we get as we divide. The pattern of remainders is made up of the six digits: 1, 3, 2, 6, 4, 5; this pattern of 6 digits is recycled, over and over again. The diagram below shows the cycle of remainders. Interestingly those remainders diametrically opposite sum to 7, our divisor.



While for example, 27 divided by 7 is usually recorded as 3 remainder 6, you can also represent this division as 4 remainder -1. Using this approach we shall replace the larger remainders by their negative complement to 7.



We can apply this now as a test for divisibility by 7. We need to remember the pattern 1, 3, 2, -1, -3, -2. These are the multipliers of the digits in our number to test for divisibility by 7.

Take the number 946 841. We start with the units digit 1 and multiply that by 1 the first number in the cycle, then 4 by 3 the second number and so on.

1 x 1 + 4 x 3 + 8 x 2 + 6 x -1 + 4 x -3 + 9 x -2

1 + 12 + 16 – 6 – 12 – 18

7

As 7 is divisible by 7 then 946 841 is divisible by 7.

Like the test of divisibility for 9, this test also involves the addition of the digits, but this time we found a weighted sum involving the cycle of 6 remainders as multipliers. There are some interesting patterns like this when you devise tests for divisibility by 11, 13 and 17. You might like to generate these for yourself.

Nigel Bufton

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CROSS NUMBER PUZZLE ANSWER FROM ISSUE 50

(1) 9	9	■	(2) 2	(3) 3	3
6	■	(4) 4	8	4	■
(5) 1	(6) 3	2	■	5	■
■	1	■	(7) 1	6	(8) 9
■	(9) 2	(10) 4	8	■	0
(11) 2	5	6	■	(12) 3	0