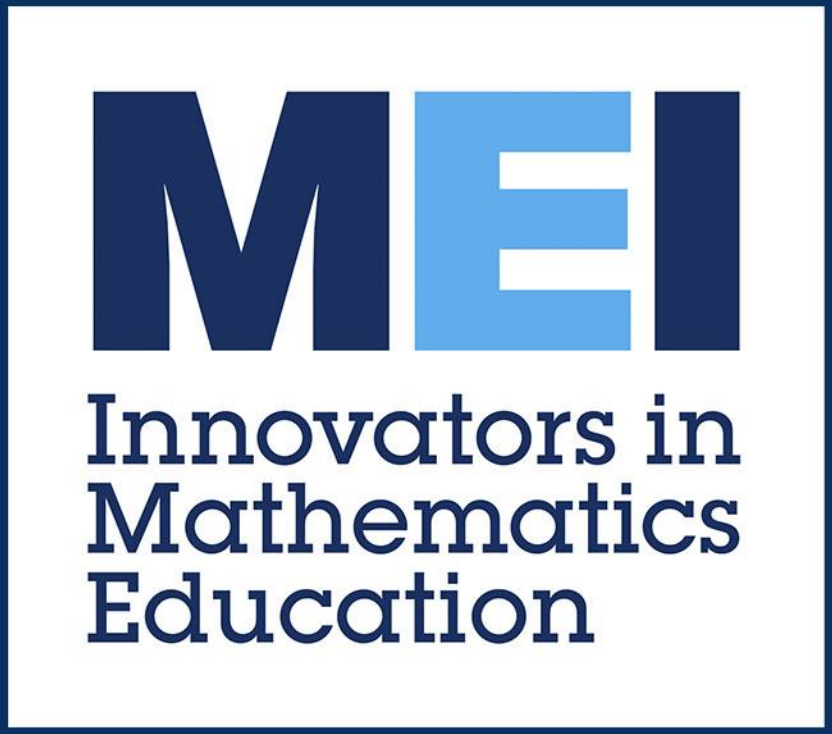

Mathematics in Education and Industry

Over 50 years at the forefront of Mathematics Education

Mechanics: a close look at two topics

Avril Steele

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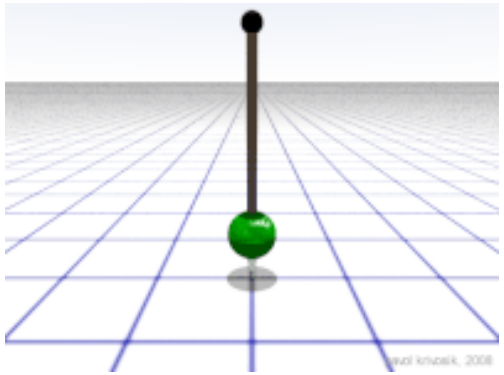
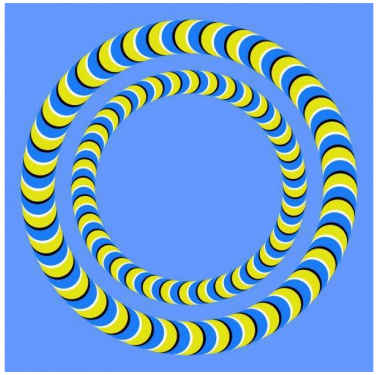
Mechanics: two topics

Dimensional analysis



Circular motion



In AS papers

Mechanics a

In A level papers

Mechanics minor

Mechanics major

In AS paper

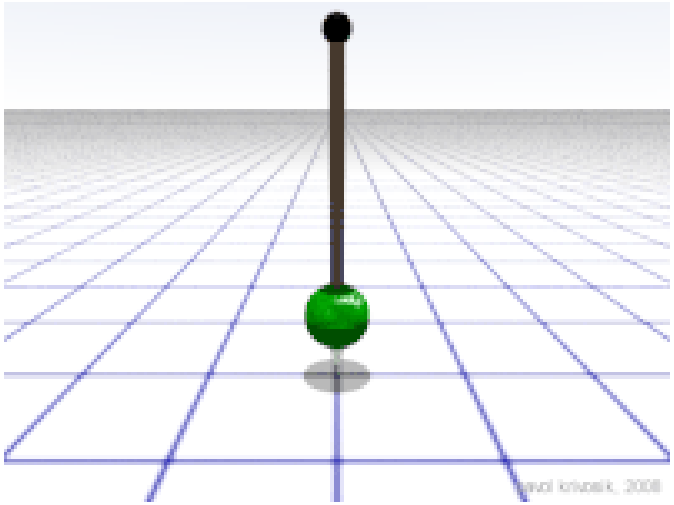
Mechanics b

In A level paper

Mechanics major

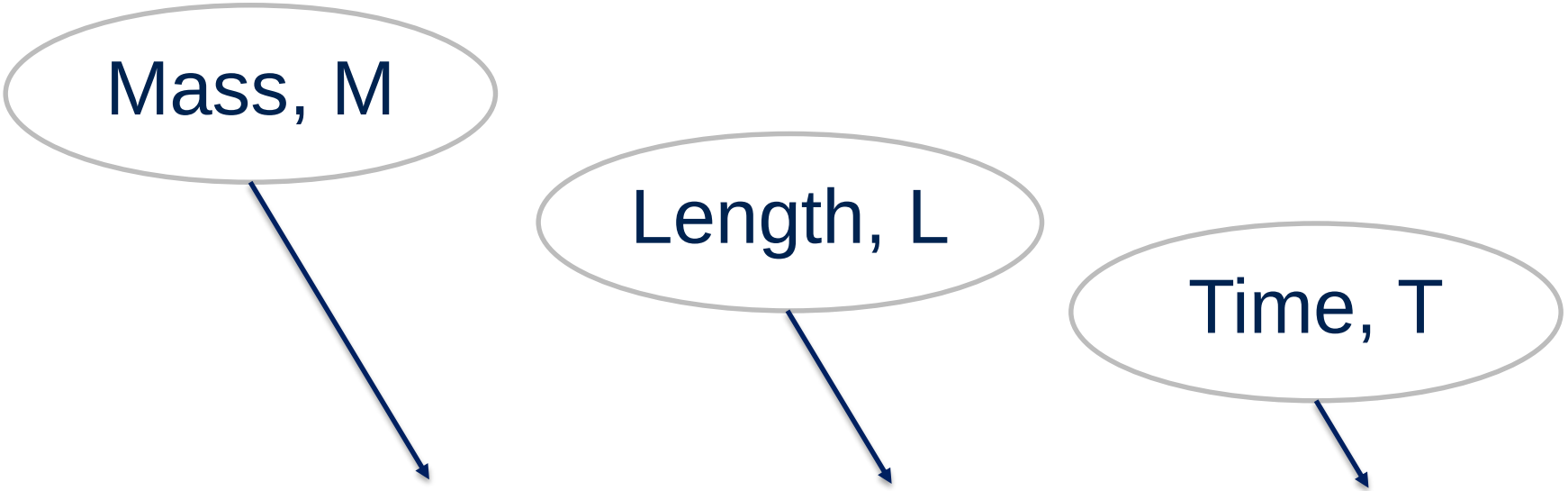
Dimensional analysis

- Dimensions of quantities & consistency
- Use in modelling



Dimensions of quantities

- Any quantity in mechanics can be expressed in terms of the following 3 fundamental dimensions



- SI units: kg m s

Dimensions of quantities

What are the dimensions of velocity and acceleration?

$$[\text{Velocity}] = LT^{-1}$$

$$[\text{Acceleration}] = LT^{-2}$$

Dimensions of quantities

Know the dimensions of angle and frequency.

Work out the dimensions of density, pressure and other quantities.

Deduce the dimensions of an unfamiliar quantity.

Dimensions of quantities

- Density
- Force
- Pressure
- Energy
- Frequency
- Angles
- Angular velocity

Dimensions of quantities

- Density ML^{-3}
- Force MLT^{-2}
- Pressure $ML^{-1}T^{-2}$
- Energy $ML^{2}T^{-2}$
- Frequency T^{-1}
- Angle dimensionless
- Angular velocity T^{-1}

Dimensional consistency

Which of these could be correct formulae…?

$$Fd = mv - mu$$

$$Ft = mv - mu$$

$$Fd = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

Dimensional consistency

Which of these could be correct formulae…?

$$Fd = mv - mu$$

$$Ft = mv - mu$$

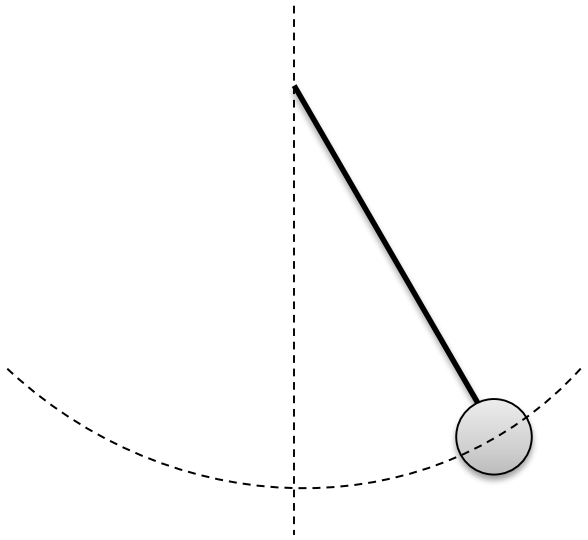
$$Fd = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

Models – determining indices

Time period of a simple pendulum.

What quantities could it depend on?

- length of string? l
- mass of bob? m
- acceleration due to gravity? g
- angle of release? θ_0



Models – determining indices

$\text{Period} = k\, l^{\alpha}\, m^{\beta}\, g^{\gamma}$ (where k is a dimensionless constant)

In terms of dimensions

$$T = L^{\alpha} M^{\beta} (LT^{-2})^{\gamma}$$

Comparing indices on each side for M, L & T

M: $\beta = 0$ T: $-2\gamma = 1 \Rightarrow \gamma = -\frac{1}{2}$ L: $\alpha + \gamma = 0 \Rightarrow \alpha = \frac{1}{2}$

Hence

$\text{Period} = k\, l^{\frac{1}{2}}\, m^{0}\, g^{-\frac{1}{2}}$ or $\text{Period} = k\sqrt{\frac{l}{g}}$

$$\text{Period} = 2\pi\sqrt{\frac{l}{g}}$$

Models – determining indices

- Formula based on a product of powers of M, L & T
- Include a dimensionless constant
- Use indices to compare powers of M, L & T

From the SAMS

3 **(i)** Find the dimensions of density and pressure (force per unit area). **[2]**

The frequency, f, of the note emitted by an air horn is modelled as $f = ks^{\alpha} p^{\beta} d^{\gamma}$, where

- s is the length of the horn,
- p is the air pressure,
- d is the air density,
- k is a dimensionless constant.

(ii) Determine the values of α, β and γ. **[4]**

A particular air horn emits a note at a frequency of 512 Hz and the air pressure and air density are recorded. At another time it is found that the air pressure has fallen by 2% and the air density has risen by 1%. The length of the horn is unchanged.

(iii) Calculate the new frequency predicted by the model. **[2]**

Use a model

Draft

frequency, $f = ks^{\alpha}p^{\beta}d^{\gamma}$

solution, $f = ks\sqrt{\frac{p}{d}}$

- s is the length of the horn,
- p is the air pressure,
- d is the air density,
- k is a dimensionless constant.

- A note of frequency 512Hz is recorded along with p and d.
- At another time p has fallen by 2% and d has risen by 1%.
- Calculate the new frequency predicted by the model.

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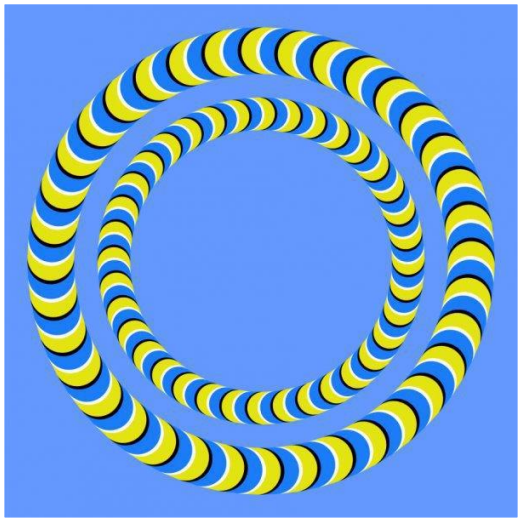
- A note of frequency 512Hz is recorded along with p and d.
- At another time p has fallen by 2% and d has risen by 1%.
- Calculate the new frequency predicted by the model.

$$\frac{f_2}{f_1} = \frac{\cancel{ks}\sqrt{\frac{p_2}{d_2}}}{\cancel{ks}\sqrt{\frac{p_1}{d_1}}} \qquad \frac{f_2}{512} = \frac{\sqrt{\frac{0.98\cancel{p_1}}{1.01\cancel{d_1}}}}{\sqrt{\frac{\cancel{p_1}}{\cancel{d_1}}}} \qquad f_2 = 512 \times \sqrt{\frac{0.98}{1.01}}$$

$$f_2 = 504\ Hz$$

Circular motion

- Force & acceleration
- Motion with uniform speed
- Motion with non-uniform speed



Circular motion

- Understand the language associated with circular motion.
- Identify the force(s) acting on a body in circular motion.
- Be able to calculate acceleration towards the centre of circular motion.
- Be able to model situations involving circular motion with uniform speed in a horizontal plane.
- Be able to model situations involving circular motion with non-uniform speed.
- Be able to calculate tangential acceleration.
- Be able to model situations involving motion in a vertical circle.
- Identify the conditions under which a particle departs from circular motion.

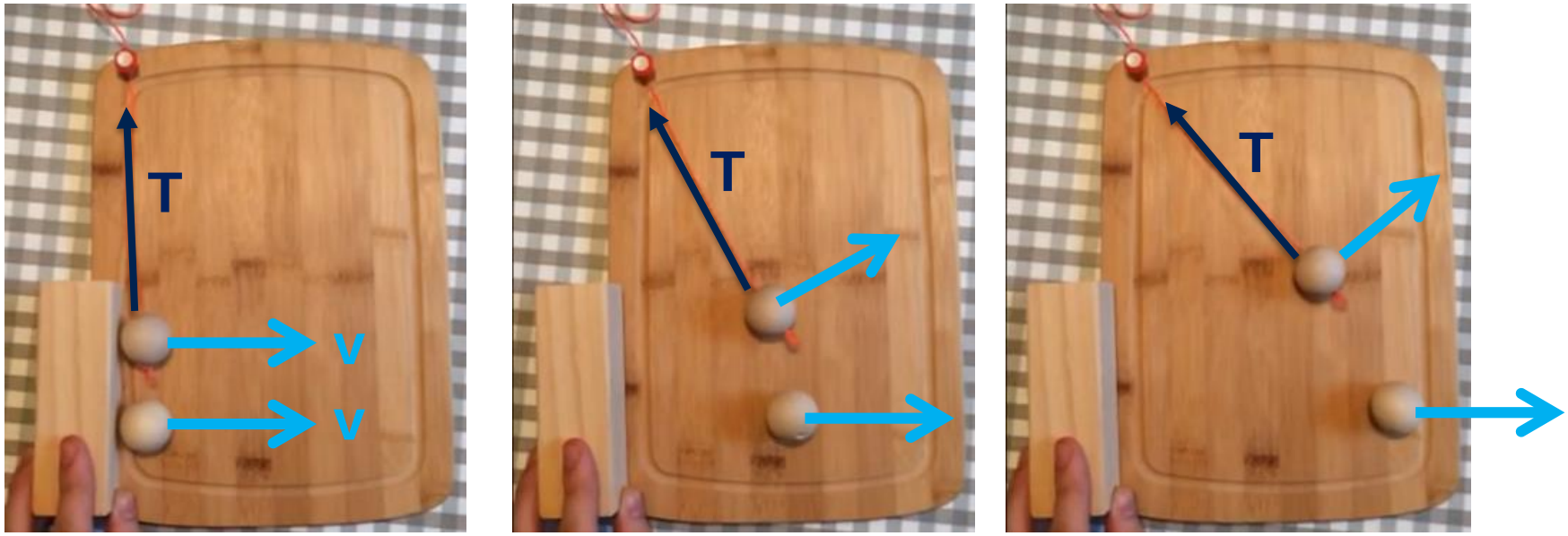
Circular motion

What makes objects move in circles?

- Why does the moon circle the earth?
- What happens to the ‘hammer’ when the athlete lets it go?
- Does the pilot of the plane need to be strapped into her seat at the top of a loop in order not to fall out?

What causes circular motion?

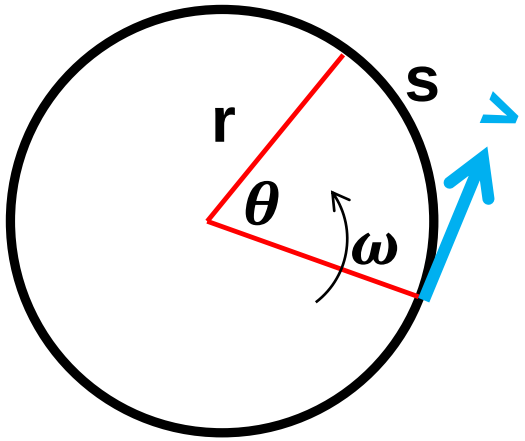
What causes circular motion?



A change in velocity means…

… a force must be acting.

Connecting linear and circular motion



The angular velocity, ω (omega), is the rate at which θ is increasing (measured in radians per second).

$$\omega = \frac{d\theta}{dt}$$

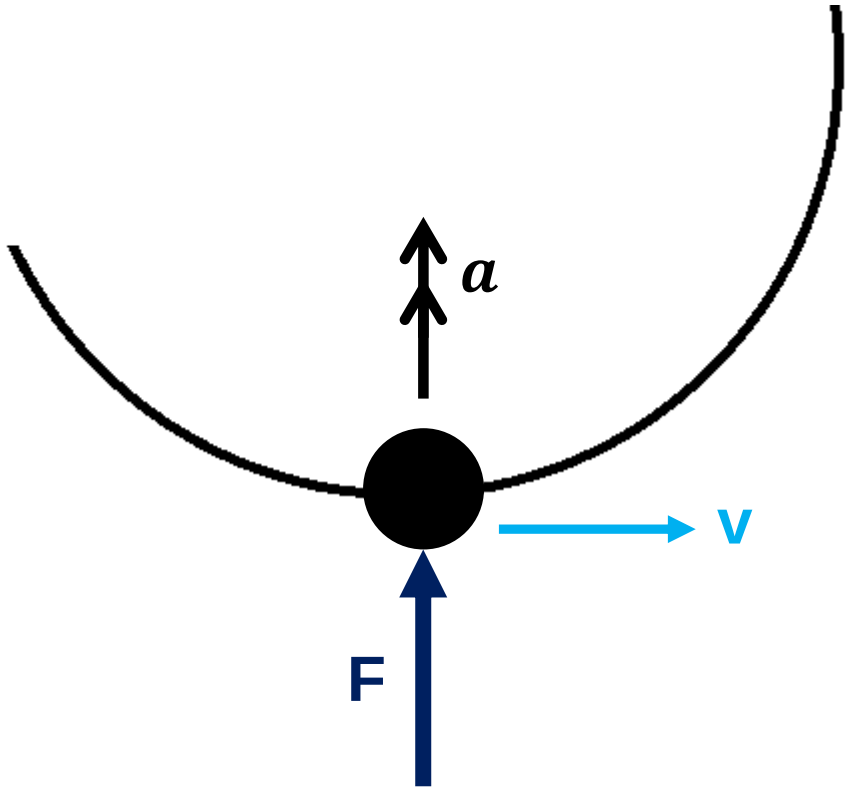
$$s = r\theta$$

$$v = \frac{ds}{dt} = \frac{d(r\theta)}{dt} = r\frac{d\theta}{dt} = r\omega$$

For a particle moving in a circle of radius *r*, the relationship between the angular velocity ω and the speed v is given by

$$v = r\omega \quad \text{or} \quad \omega = \frac{v}{r}$$

Motion in a circle (constant speed)

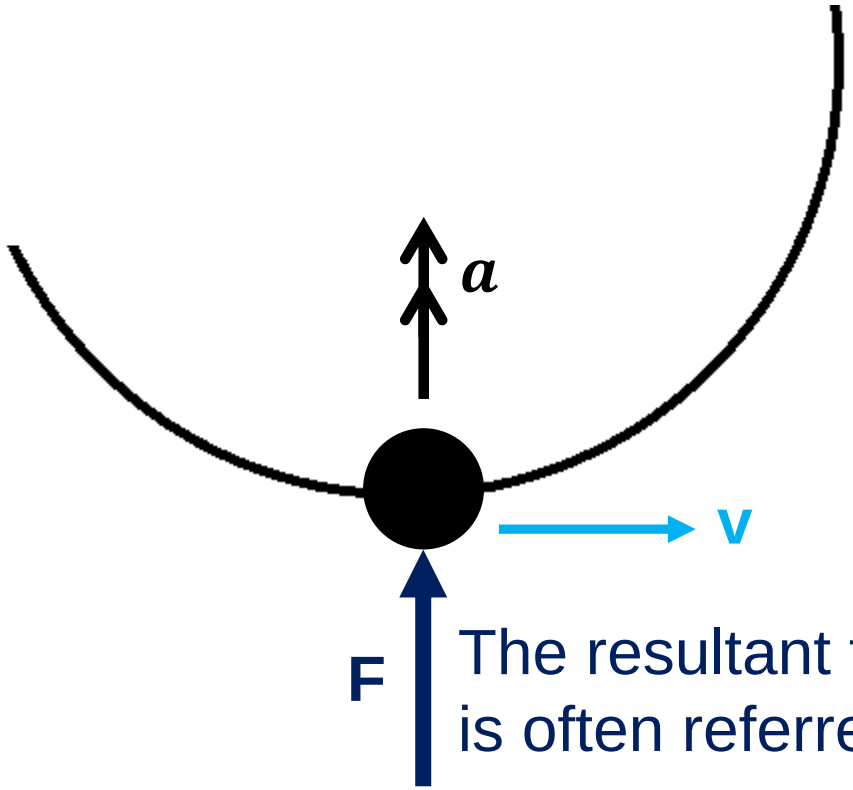


Consider an object moving at a constant speed *v*, in a circle of radius r.

As there is no change in speed there is no force component that acts in the direction of motion.

Therefore the force must act along the radius towards the centre of the circle producing a radial acceleration.

Motion in a circle (constant speed)



It can be shown that the radial acceleration is related to the radius and the angular velocity by:

$$a = r\omega^2$$

or

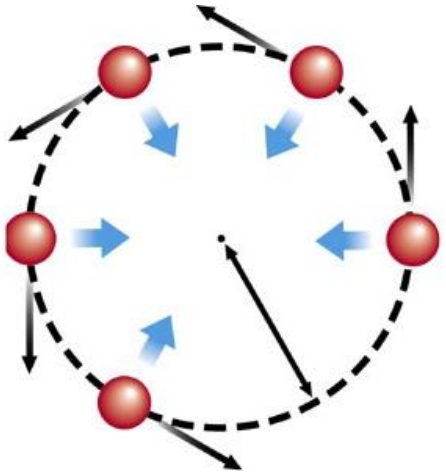
$$a = \frac{v^2}{r}$$

The resultant force causing this acceleration is often referred to as the centripetal force.

Centripetal means ‘centre-seeking’.

Force & acceleration

- The important point to remember here is that this force must come from somewhere - an object will not move in a circle unless there is a resultant force acting on it towards the centre of the circle.



- Newton's 2nd law will be used to relate resultant force to acceleration and solve problems.

Force & acceleration

Examples of forces might include:

- **tension** in a string from a particle to the centre of the circle.
- **friction** between an object and a rotating platform. If there were no friction, the object would move in a straight line and fly off the platform. Friction acts against this motion and therefore acts towards the centre of the circle.
- **reaction** force between an object and some kind of barrier (e.g. for a toy train going round a curve, the edges of the tracks give a horizontal reaction force).
- For a banked circular track, a **component of the normal reaction** force between vehicle and track acts towards the centre of the circle.

1. String - Tension

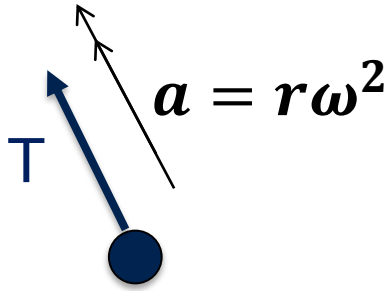
2. Rotating platform - Friction

3. Edge or barrier - Reaction

4. Sloping surface - Reaction

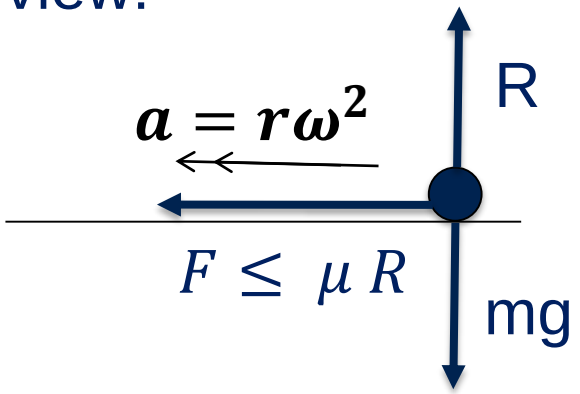
1. String – Tension

Plan view:



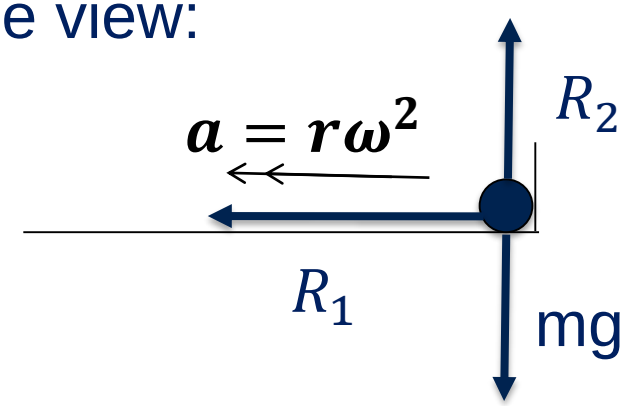
2. Rotating platform – Friction

Side view:



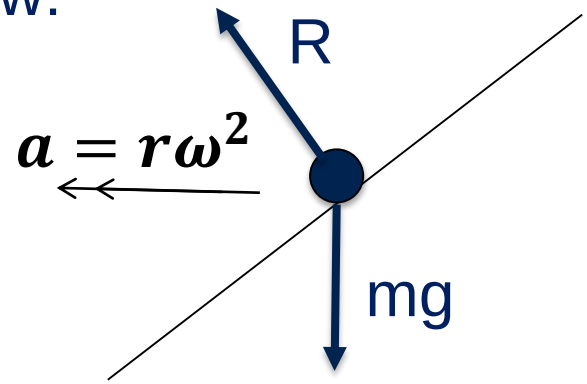
3. Edge or barrier - Reaction

Side view:



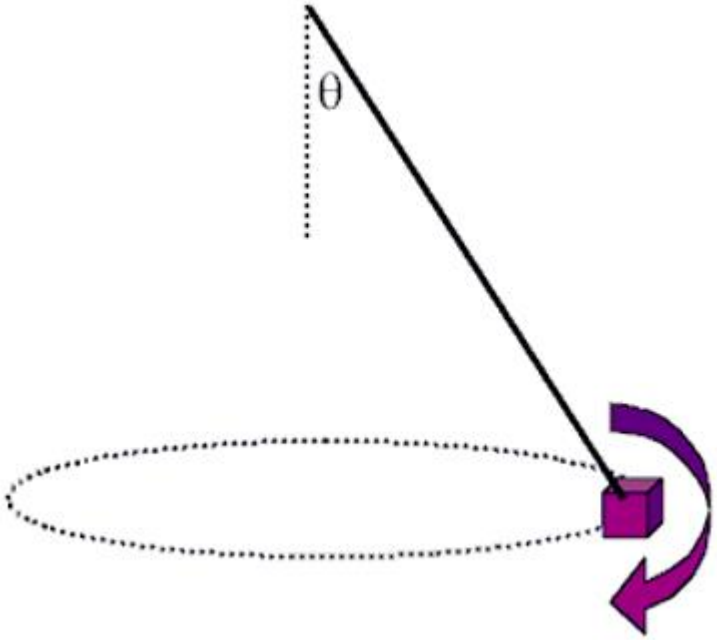
4. Sloping surface - Reaction

Side view:



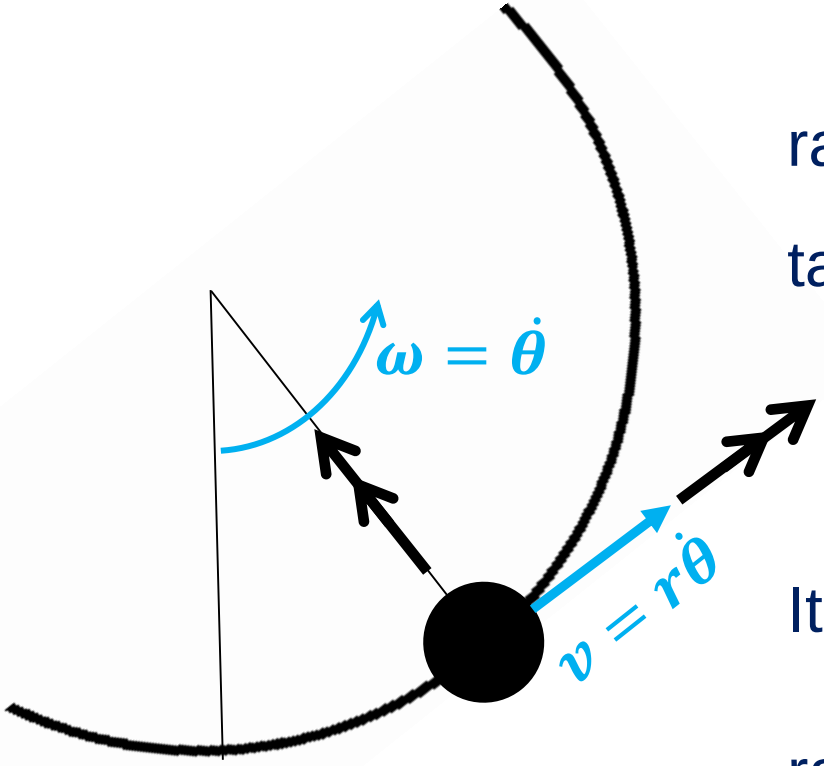
Motion in a horizontal plane

- A conical pendulum



Resource courtesy of
http://furthermaths.org.uk/teaching-mechanics

Motion in a circle (non-constant speed)



radial velocity = 0

tangential velocity = $v = r\omega = r\dot{\theta}$

It can be shown that:

radial acceleration = $r\omega^2 = \frac{v^2}{r}$

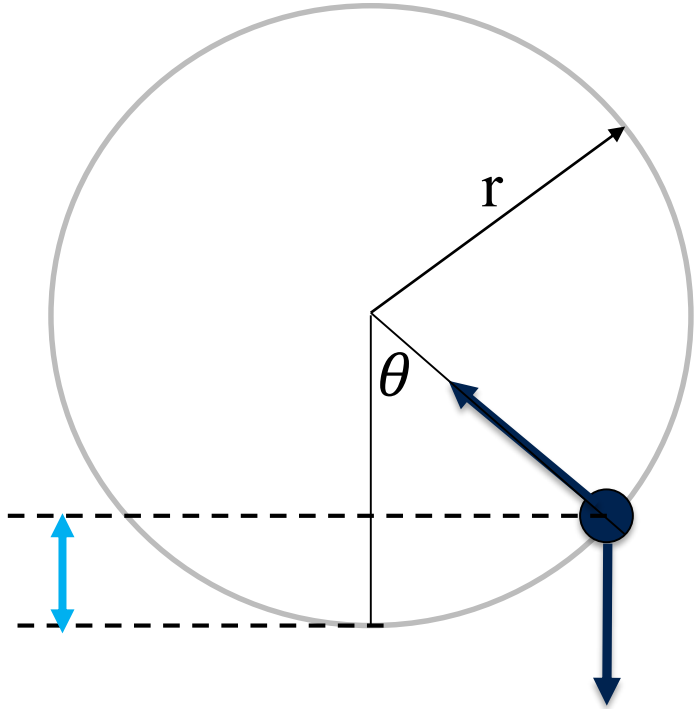
tangential acceleration = $r\ddot{\theta}$

Motion in a vertical circle

- draw force/acceleration diagram
- apply Newton's 2nd Law radially
- use conservation of energy

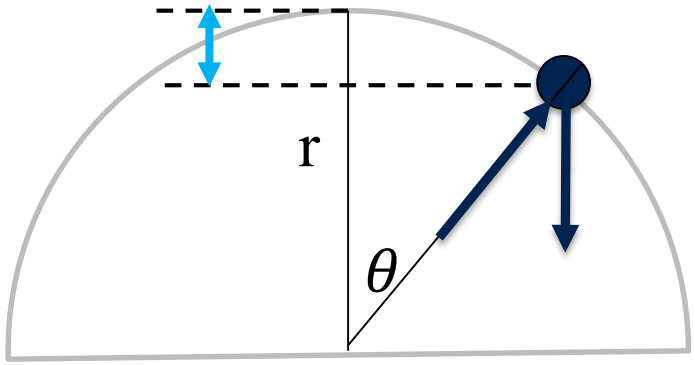
- conditions for object to leave circular path
  - string becomes slack (zero tension)
  - contact reaction becomes zero

1. Mass on string/rod/inside a smooth track.



height difference = $r - rcos\theta$

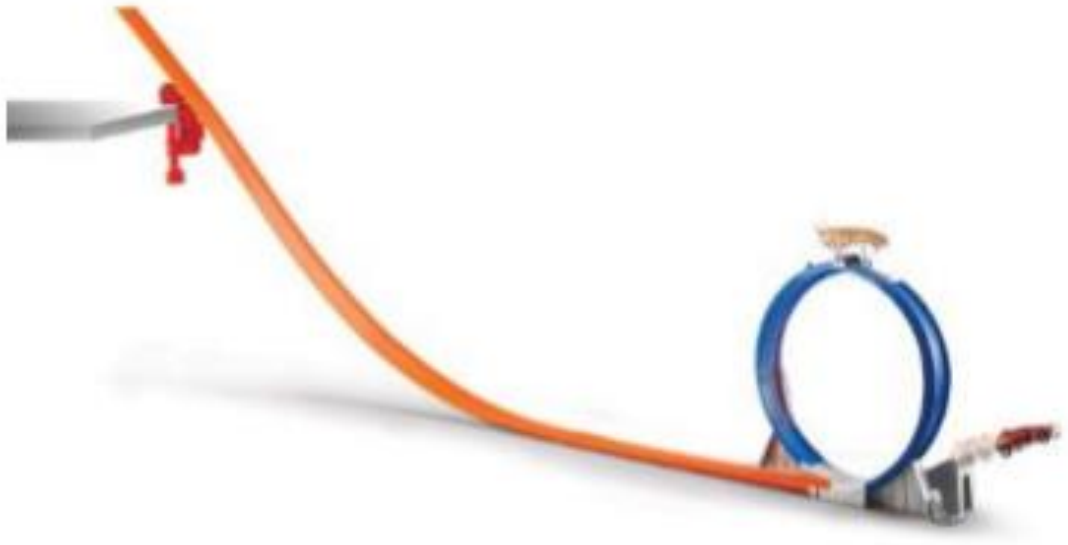
2. Motion on the outside of a smooth cylinder/hemisphere.



height difference = $r - rcos\theta$

Motion in a vertical circle

- Hot wheels loop the loop!



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Finally…

- So what about ‘centrifugal force’?
  – this means ‘centre-fleeing’ force.
- Why is this so confusing to us?
- Have you ever felt thrown outwards whilst travelling in a circle?
- Does centrifugal force exist?

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- We also pioneer the development of innovative teaching and learning resources