The Pendulum Plain and Puzzling

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April 2017

Outline

Connections within and beyond mathematics

The pendulum is a paradigm.

1. Almost every interesting dynamic phenomena can be illustrated by a pendulum.

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    3. Chaos

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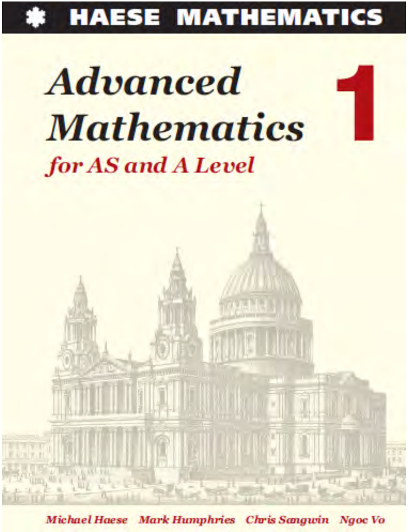
Illustrated by examples.
(Equations on request!)

Mechanics within mathematics



The elastic pendulum

What happens when I displace vertically and release from rest?

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$$y(t) = A\cos(kt + \rho).$$

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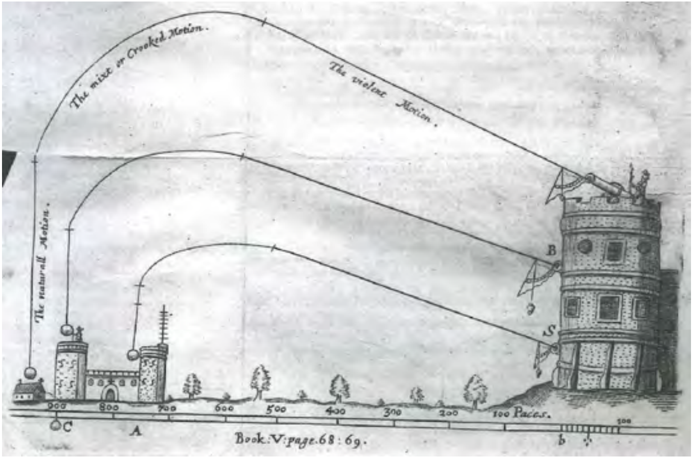
Where is the x-coordinate in the model?

Modelling

I hope I shall shock a few people in asserting that the most important single task of mathematical instruction in the secondary school is to teach the setting up of equations to solve word problems. [...] And so the future engineer, when he learns in the secondary school to set up equations to solve “word problems" has a first taste of, and has an opportunity to acquire the attitude essential to, his principal professional use of mathematics.
Polya (1962)

Pre-Newtonian mechanics


The mixt or Crooked Motion.
The violent Motion.
The naturall Motion.
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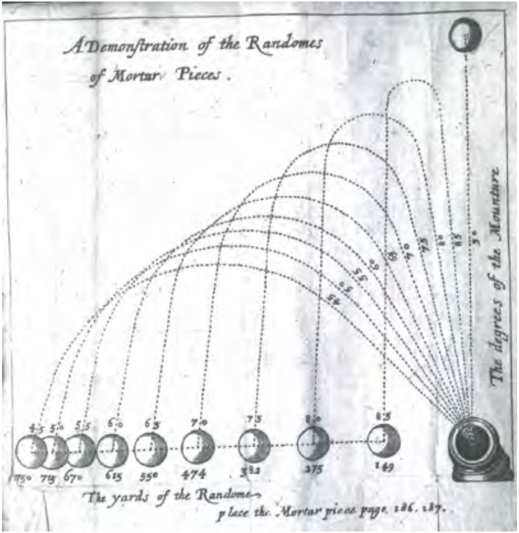
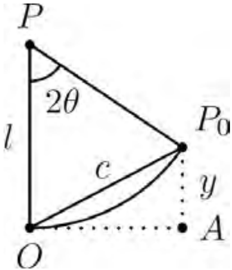
Randomness

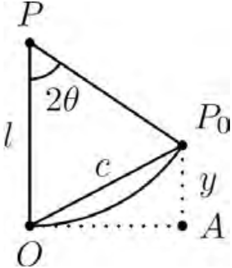


A Demonſtration of the Randomes of Mortar Pieces.
The degrees of the Mountura
The yards of the Randome
place the Mortar piece page. 186. 187.

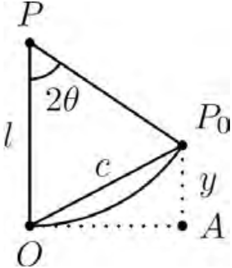
A proposition well known to geometers

... it is a proposition well known to geometers, that the velocity of a pendulous body in the lowest point is as the chord of the arc which it has described in its descent. (Newton, Principia, (I), pg 25)





$$y = l(1 - \cos(2\theta)) = 2l\sin^2(\theta).$$



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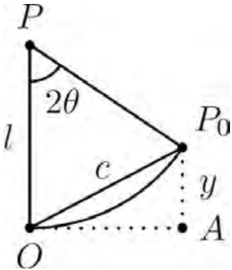
By conservation of energy, the kinetic energy at O equals the potential energy at P_0.



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$$gy = 2gl\sin^2(\theta) = \frac{1}{2}v^2.$$

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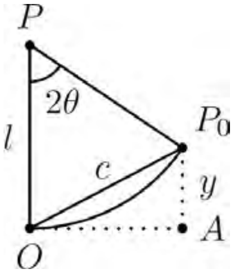

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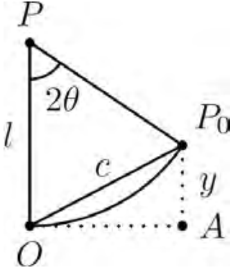

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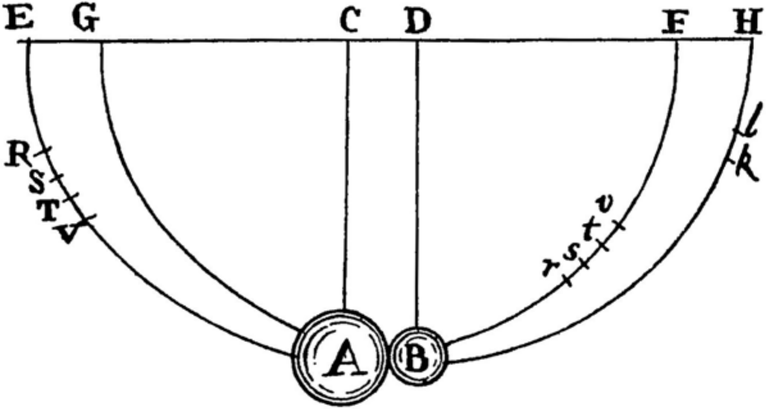
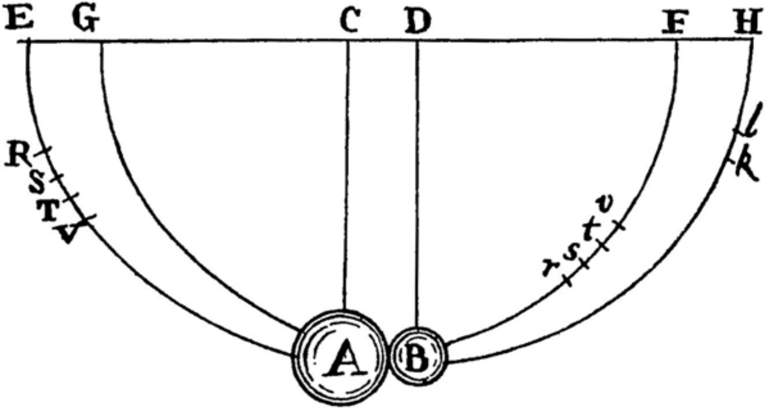
MEI AS FM 6.02e

The pendulum as experimental apparatus



MEI FM 6.03e

The pendulum as experimental apparatus



MEI FM 6.03e

Issac Newton

1643 – 1727

Newton's Laws of Motion

- Every object in a state of uniform motion tends to remain in that state of motion unless an external force is applied to it.
- The relationship between an object's mass m, its acceleration a, and the applied force F is

$$F = ma.$$

- For every action there is an equal and opposite reaction.

AS/A level core

Simple harmonic motion

Point at P. Note $s = l\theta$ and $a = l\ddot{\theta}$.

$$ml\ddot{\theta} = -mg\sin(\theta), \quad \theta(0) = \theta^0,\ \dot{\theta}(0) = \dot{\theta}^0.$$

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Then the solution is

$$\theta(t) = A\cos(\omega t + \rho).$$

where $\omega^2 = g/l$

Notes on SHM

- Many assumptions: e.g. ignore friction.

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  For a real pendulum it does.
- The Hooke's law spring/mass gives SHM.

MEI AS FM 4.10f

Christiaan Huygens

1629 – 1695

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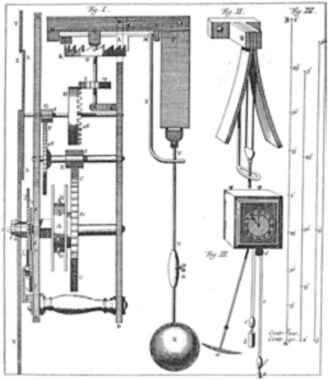
Horologium oscillatorium, (1673)

The Pendulum Clock or Geometrical Demonstrations Concerning the Motion of Pendula As Applied to Clocks



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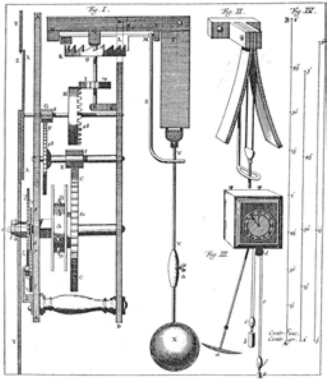

A pendulum which changes its length (cf elastic pendulum)

Energy considerations (simple pendulum)

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$$\frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos(\theta) = E_0,$$

or

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Kinetic and potential energy.

MEI FM 6.02

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Conservation equation: E_1, is constant

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Conservation equation: E_1, is constant

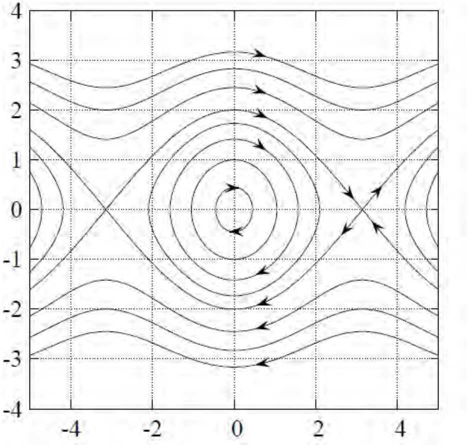
Writing this as

$$\dot{\theta}^2 = \frac{2g}{l}(\cos(\theta) - 1) + \frac{2E_1}{ml^2},$$

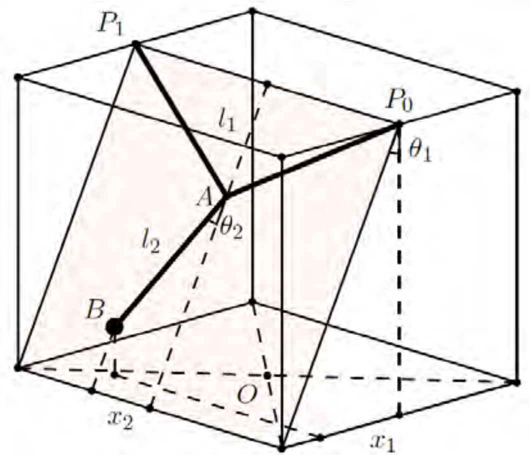
The form $y^2 = \alpha(\cos(x) - 1) + \beta$ where $\alpha, \beta > 0$.

Phase plane

$y^2 = \alpha(\cos(x) - 1) + \beta$



Blackburn Y pendulum



Blackburn Y pendulum

For small oscillations

$$x_1(t) = A_1 \cos(\omega_1 t + \rho_1).$$

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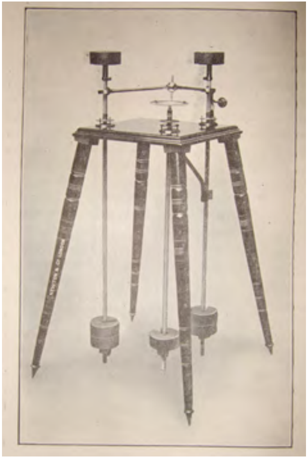
Lissajous figure

Add in damping



Coupled oscillators

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Wilberforce's spring (1894)

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$$I\ddot{\phi} = -\tau\phi - \frac{1}{2}\epsilon y.$$

These are weakly coupled linear oscillators.

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$$\omega_s^2 = \frac{k}{m}, \quad \omega_t^2 = \frac{\tau}{I}.$$

Techniques

Many methods:

- Direct substitution of candidate $y(t) = e^{\omega t}$.
- Set up whole system as $\dot{x} = Ax$.
- Laplace transforms

Solution

After some work ...

$$2\omega^2 = \left(\omega_s^2 + \omega_t^2\right) \pm \sqrt{\left(\omega_s^2 - \omega_t^2\right)^2 + \epsilon^2/(ml)}. \qquad (1)$$

and

$$\begin{aligned} \phi(t) &= \frac{\epsilon y_0}{2l(\omega_F^2 - \omega_S^2)} \left(\cos(\omega_F t) - \cos(\omega_S t)\right), \\ y(t) &= \frac{y_0}{l(\omega_F^2 - \omega_S^2)} \left((l\omega_F^2 - \tau)\cos(\omega_F t) - (l\omega_S^2 - \tau)\cos(\omega_S t)\right). \end{aligned}$$

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\end{aligned}$$

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which is the product of two sinusoids.

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which is the product of two sinusoids.
Special case ($\omega_F = \omega_S$) when

$$\omega_s = \omega_t$$

complete energy transfer between torsion and vertical modes.

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complete energy transfer between torsion and vertical modes.
Physics experiment to measure the value of τ.

Elastic pendulum

I first noticed the mode coupling in the swinging pendulum when I was a Teaching Assistant as a graduate student when (by chance) a student could not make the lab experiment work. This classical experiment, as you know, first measures the spring constant and then predicts the period. After stewing about this for a while I was able to work it out. (Olsson, M. 2008, private communication)

See Olsson 1976.

Deriving equations

Kinetic energy

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2), \tag{2}$$

Potential energy

$$V = mgy + \frac{k}{2}(l - l_0)^2, \tag{3}$$

$$l^2 = (l_1 - y)^2 + x^2.$$

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Approximate the Lagrangian $L = T - V$ as

Approximate equations of motion

$$\ddot{x} = -\omega_p^2 x + \lambda xy,$$
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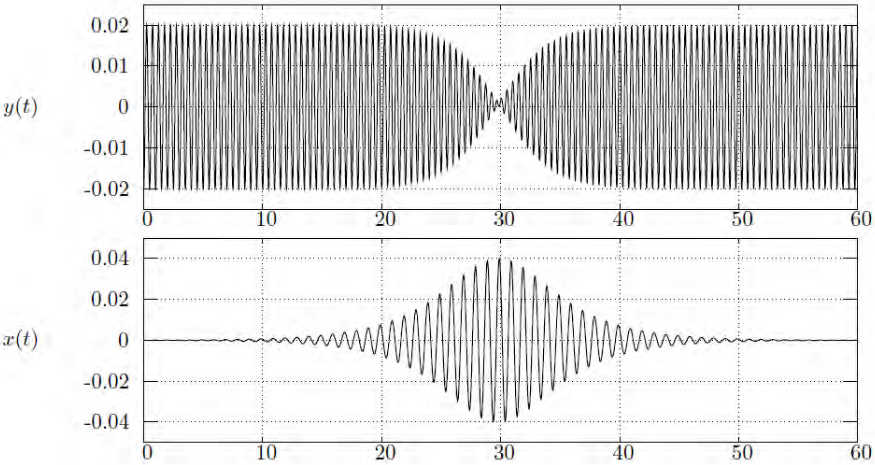
where

$$\omega_p^2 = \frac{g}{l_1}, \quad \omega_s^2 = \frac{k}{m}$$
$$\lambda = \omega_s^2 \frac{l_0}{l_1^2} = \frac{k}{m}\frac{l_0}{l_1^2}.$$

Numerical solutions



Start with a vertical motion

$|x| << 1$

$$y(t) \approx a\cos(\omega_s t).$$

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So

$$\ddot{x}(t) + (\omega_p^2 + \lambda a\cos(\omega_s t))x(t) = 0.$$

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Mathieu's equation

$\omega_s = 2\omega_p$.

Mathieu's equation

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- Instability of the elastic pendulum
- Stability of the driven inverted pendulum

A pendulum theorem†

BY D. J. ACHESON
Jesus College, Oxford OX1 3DW, U.K.

We consider N linked pendulums which are inverted and balanced on top of one another, and establish a general theorem which shows how they may be stabilized by small vertical oscillations of the support.

Driven Double Pendulum

Chaos

Simple systems:

- simple behavior.

Chaos

Simple systems:

- simple behavior.
- complex behavior!

Chaos

Simple systems:

- simple behavior.
- complex behavior!

Double pendulum.

Chaos

Simple systems:

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Double pendulum.
Complexity very different from irrational Lissajous.

Magnetic pendulum

Release the pendulum from rest.
Which magnet does it end up above?

Magnetic pendulum

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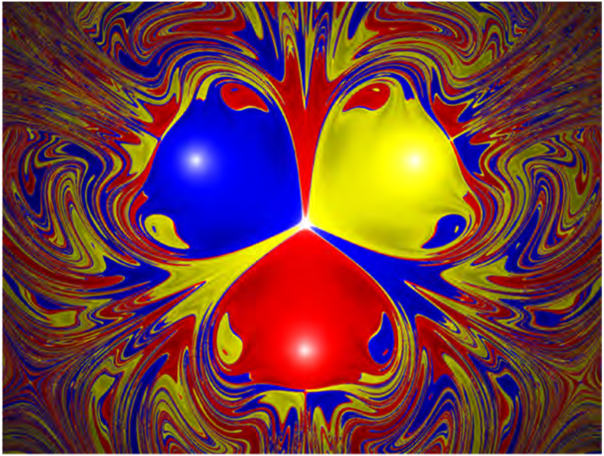

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