The Pendulum Plain and Puzzling

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Outline

1. Introduction and motivation
2. Geometrical physics
3. Classical mechanics
4. Driven pendulum
5. The elastic pendulum
6. Chaos
7. Conclusion
Connections within and beyond mathematics

The pendulum is a paradigm.

1. Almost every interesting dynamic phenomena can be illustrated by a pendulum.
Connections within and beyond mathematics

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2. The pendulum illustrates the concerns of each age.
   1. Divinely ordered world of classical mechanics
   2. Catastrophe theory
   3. Chaos
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3. Contemporary examples relevant to A-level mathematics/FM.
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Illustrated by examples.
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Illustrated by examples. (Equations on request!)
Mechanics within mathematics
The elastic pendulum

What happens when I displace vertically and release from rest?

\[ y = -k \frac{1}{2} y^2 \]

So \( y(t) = A \cos(kt + \varphi) \).

Where is the \( x \)-coordinate in the model?
The elastic pendulum

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Modelling

I hope I shall shock a few people in asserting that the most important single task of mathematical instruction in the secondary school is to teach the setting up of equations to solve word problems. [...] And so the future engineer, when he learns in the secondary school to set up equations to solve "word problems" has a first taste of, and has an opportunity to acquire the attitude essential to, his principal professional use of mathematics.
Polya (1962)
Pre-Newtonian mechanics
A proposition well known to geometers

... *it is a proposition well known to geometers, that the velocity of a pendulous body in the lowest point is as the chord of the arc which it has described in its descent.* (Newton, *Principia*, (I), pg 25)
\[ y = l(1 - \cos(2\theta)) = 2l \sin^2(\theta). \]
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\[ gy = 2gl \sin^2(\theta) = \frac{1}{2} v^2. \]

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Hence \( v = \sqrt{\frac{g}{l} c}. \)
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The pendulum as experimental apparatus
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Issac Newton
1643 – 1727
Newton’s Laws of Motion

- Every object in a state of uniform motion tends to remain in that state of motion unless an external force is applied to it.
- The relationship between an object’s mass \( m \), its acceleration \( a \), and the applied force \( F \) is
  \[
  F = ma.
  \]
- For every action there is an equal and opposite reaction.

AS/A level core
Simple harmonic motion

Point at $P$. Note $s = l\theta$ and $a = l\ddot{\theta}$.

$$ml\ddot{\theta} = -mg \sin(\theta), \quad \theta(0) = \theta^0, \quad \dot{\theta}(0) = \dot{\theta}^0.$$
Simple harmonic motion

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For small $\theta$ assume $\sin(\theta) \approx \theta$. 

\[ ml\ddot{\theta} = -mg\theta. \]

Then the solution is $\theta(t) = A\cos(\omega t + \rho)$.

where $\omega^2 = \frac{g}{l}$. 

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\[\text{Pendulum}\]  
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Physics experiment to establish value of \( g \).
Notes on SHM

- Many assumptions: e.g. ignore friction.
- $\sin(\theta) \approx \theta$ makes it easier.
- Physics experiment to establish value of $g$.
- For SHM the period does not depend on amplitude.
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Physics experiment to establish value of \( g \).

For SHM the period does not depend on amplitude. For a real pendulum it does.
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The Hooke’s law spring/mass gives SHM.
Christiaan Huygens
1629 – 1695
The Pendulum Clock or Geometrical Demonstrations Concerning the Motion of Pendula As Applied to Clocks
Christiaan Huygens
Horologium oscillatorium, (1673)

The Pendulum Clock or Geometrical Demonstrations Concerning the Motion of Pendula As Applied to Clocks

A pendulum which changes its length (cf elastic pendulum)
Energy considerations (simple pendulum)

\[ ml\ddot{\theta} = -mg \sin(\theta), \quad \theta(0) = \theta_0, \quad \dot{\theta}(0) = \dot{\theta}_0. \]
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Multiply by \( l\dot{\theta} \) so that,

\[ ml^2 \dot{\theta} \ddot{\theta} + mgl \dot{\theta} \sin(\theta) = 0. \]
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Integrating this gives

\[ \frac{1}{2} ml^2 \dot{\theta}^2 - mgl \cos(\theta) = E_0, \]

or

\[ \frac{1}{2} ml^2 \dot{\theta}^2 + mgl(1 - \cos(\theta)) = E_1. \]
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Kinetic and potential energy.
Energy considerations

\[ \frac{1}{2} m l^2 \dot{\theta}^2 + m g l (1 - \cos(\theta)) = E_1. \]

*Conservation equation:* \( E_1 \), is constant
Energy considerations

\[ \frac{1}{2} ml^2 \dot{\theta}^2 + mgl(1 - \cos(\theta)) = E_1. \]

Conservation equation: \( E_1 \), is constant
Writing this as
\[ \dot{\theta}^2 = \frac{2g}{l}(\cos(\theta) - 1) + \frac{2E_1}{ml^2}, \]

The form \( y^2 = \alpha(\cos(x) - 1) + \beta \) where \( \alpha, \beta > 0 \).
Phase plane

\[ y^2 = \alpha (\cos(x) - 1) + \beta \]
Blackburn Y pendulum
Blackburn Y pendulum

For small oscillations

\[ x_1(t) = A_1 \cos(\omega_1 t + \rho_1). \]

\[ x_2(t) = A_2 \cos(\omega_2 t + \rho_2). \]

If \( \omega_1 \) is rational....

Lissajous figure

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Pendulum

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Blackburn Y pendulum

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If \( \frac{\omega_1}{\omega_2} \) is rational....
Blackburn Y pendulum

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If \( \frac{\omega_1}{\omega_2} \) is rational....

Lissajous figure
Add in damping
Coupled oscillators

\[ m\ddot{y} = -ky - \frac{1}{2}\epsilon\dot{\phi}, \]

\[ I\ddot{\phi} = -\tau\dot{\phi} - \frac{1}{2}\epsilon y. \]

These are weakly coupled linear oscillators.

\[ \omega_s^2 = \frac{k}{m}, \quad \omega_t^2 = \frac{\tau}{I}. \]
Coupled oscillators

Wilberforce’s spring (1894)

\[ m \ddot{y} = -ky - \frac{1}{2} \epsilon \phi, \]
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\[ \omega_t^2 = \tau I. \]
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Techniques

Many methods:
- Direct substitution of candidate $y(t) = e^{\omega t}$.
- Set up whole system as $\dot{x} = Ax$.
- Laplace transforms
Solution

After some work ...

\[ 2\omega^2 = \left(\omega_s^2 + \omega_t^2\right) \pm \sqrt{(\omega_s^2 - \omega_t^2)^2 + \epsilon^2/(ml)}. \] (1)

and

\[ \phi(t) = \frac{\epsilon y_0}{2I(\omega_F^2 - \omega_S^2)} \left( \cos(\omega_F t) - \cos(\omega_S t) \right), \]

\[ y(t) = \frac{y_0}{l(\omega_F^2 - \omega_S^2)} \left( (l\omega_F^2 - \tau) \cos(\omega_F t) - (l\omega_S^2 - \tau) \cos(\omega_S t) \right). \]
Solution

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\[ 2\omega^2 = \left( \omega_S^2 + \omega_t^2 \right) \pm \sqrt{\left( \omega_S^2 - \omega_t^2 \right)^2 + \epsilon^2 / (ml)}. \]  

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which is the product of two sinusoids.
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Special case \((\omega_F = \omega_S)\) when

\[ \omega_S = \omega_t \]

complete energy transfer between torsion and vertical modes.
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complete energy transfer between torsion and vertical modes.
Physics experiment to measure the value of \( \tau \).
I first noticed the mode coupling in the swinging pendulum when I was a Teaching Assistant as a graduate student when (by chance) a student could not make the lab experiment work. This classical experiment, as you know, first measures the spring constant and then predicts the period. After stewing about this for a while I was able to work it out. (Olsson, M. 2008, private communication)

See Olsson 1976.
Deriving equations

Kinetic energy

\[ T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2), \quad (2) \]

Potential energy

\[ V = mgy + \frac{k}{2}(l - l_0)^2, \quad (3) \]

\[ l^2 = (l_1 - y)^2 + x^2. \]
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Approximate the Lagrangian \( L = T - V \) as
Approximate equations of motion

\[ \ddot{x} = -\omega_p^2 x + \lambda xy, \]
\[ \ddot{y} = -\omega_s^2 y + \frac{\lambda}{2} x^2. \]
Approximate equations of motion

\[ \ddot{x} = -\omega^2_p x + \lambda xy, \]
\[ \ddot{y} = -\omega^2_s y + \frac{\lambda}{2} x^2. \]

where

\[ \omega^2_p = \frac{g}{l_1}, \quad \omega^2_s = \frac{k}{m} \]
\[ \lambda = \omega^2_s \frac{l_0}{l_1^2} = \frac{k}{m} \frac{l_0}{l_1^2}. \]
Numerical solutions
Start with a vertical motion

\[ |x| \ll 1 \]

\[ y(t) \approx a \cos(\omega_s t). \]
Start with a vertical motion

$|x| \ll 1$

$$y(t) \approx a \cos(\omega_s t).$$

So

$$\ddot{x}(t) + (\omega_p^2 + \lambda a \cos(\omega_s t))x(t) = 0.$$
Start with a vertical motion

$|x| \ll 1$

$y(t) \approx a \cos(\omega_s t)$. 

So

$\ddot{x}(t) + (\omega_p^2 + \lambda a \cos(\omega_s t))x(t) = 0.$

Mathieu’s equation

$\omega_s = 2\omega_p$. 
Mathieu’s equation

\[ \ddot{x}(t) + (\alpha + \beta \cos(t))x(t) = 0 \]
Mathieu’s equation

\[ \ddot{x}(t) + (\alpha + \beta \cos(t))x(t) = 0 \]

- Instability of the elastic pendulum
- Stability of the driven inverted pendulum

**A pendulum theorem†**

*By D. J. Acheson*

*Jesus College, Oxford OX1 3DW, U.K.*

We consider \(N\) linked pendulums which are inverted and balanced on top of one another, and establish a general theorem which shows how they may be stabilized by small vertical oscillations of the support.
Driven Double Pendulum
Chaos

Simple systems:
- simple behavior.
Chaos

Simple systems:
- simple behavior.
- complex behavior!
Chaos

Simple systems:
- simple behavior.
- complex behavior!

Double pendulum.
Chaos

Simple systems:

- simple behavior.
- complex behavior!

Double pendulum.
Complexity very different from irrational Lissajous.
Magnetic pendulum

Release the pendulum from rest. Which magnet does it end up above?
Magnetic pendulum

Release the pendulum from rest. Which magnet does it end up above?
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