How to prove it (or not)

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My favourite maxim

- It is better to solve one problem in five different ways than to solve five problems using the same method

- Today I will suggest how to do this in the classroom

My favourite maxim

- It is better to solve one problem in five different ways than to solve five problems using the same method

- I will do this in the context of a familiar problem:
  - Why is $\sqrt{2}$ irrational?

Analysing the problem

- Why is $\sqrt{2}$ irrational?

- How can we tackle this problem?
- The first thing is to understand it
- What does ‘irrational’ mean?
- A number is irrational if it cannot be expressed as a fraction with whole number parts

Analysing the problem

- Why is √2 irrational?

- We must show that it cannot be expressed as a fraction with whole number parts
- What is the interesting word here?
- The interesting word is ‘cannot’
- What might we do if this were replaced by ‘can’?

Analysing the problem

- Why is $\sqrt{2}$ irrational?

- What might we do if this were replaced by 'can'?
- We would find n and m so that $\sqrt{2} = \frac{n}{m}$ …
- … or so that $2m^2 = n^2$.
- But we are trying to show that this cannot be done.

Analysing the problem

- Why is √2 irrational?

- $\sqrt{2} = \frac{n}{m}$ or $2m^2 = n^2$.
- We are trying to show that this cannot be done.
- So how is this proof, if it exists, going to work?
- We try to do it and we show that something always goes wrong.
- This will be a proof by contradiction.

Tackling the problem

- Why is √2 irrational?

- So let us make the assumption that …
- $\sqrt{2} = \frac{n}{m}$ or $2m^2 = n^2$
- … and look at the consequences
- What do we know about $2m^2$?
- It is even
- So what do we know about n^2 ?
- It is even
- So what do we know about n ?
- It is even

Tackling the problem

- Why is √2 irrational?

- $\sqrt{2} = \frac{n}{m}$ or $2m^2 = n^2$.
- So we know that n is even
- So we can write $n = 2s$ …
- … and so $2m^2 = 4s^2$ or $m^2 = 2s^2$

Tackling the problem

- Why is √2 irrational?

- $\sqrt{2} = \frac{n}{m}$ or $2m^2 = n^2$.
- So $m^2 = 2s^2$
- Now we can repeat the argument, so …
- … $m = 2r$ and $2r^2 = s^2$ …
- … with r and s smaller than m and n and greater than zero

Tackling the problem

- Why is √2 irrational?

- We started with $2m^2 = n^2$…
- ... and we deduced that $2r^2 = s^2$ …
- … with r, s smaller than m, n and positive …
- … and what can we do now?
- We can do the same thing again …
- … infinitely often.

Tackling the problem

- Why is √2 irrational?

- We started with $2m^2 = n^2 \ldots$
- ... and we deduced that $2r^2 = s^2 \ldots$
- We can do the same thing infinitely often.
- Is this impossible?
- Can’t we do things infinitely often?

Tackling the problem

- Why is √2 irrational?

- We started with $2m^2 = n^2 \ldots$
- ... and we deduced that $2r^2 = s^2 \ldots$
- Can't we do things infinitely often?
- Not when these are all positive integers, and they are getting progressively smaller
- If they were positive fractions or negative integers, there would not be a problem here!
- So this is our proof by contradiction

Tackling the problem

- Why is $\sqrt{2}$ irrational?

- So this is our proof by contradiction
- Note that I have deliberately phrased this in the form of ‘doing something infinitely often’ rather than starting with an assumption of coprimality/lowest terms
- I think this is more honest and it does not presuppose how the proof is going to work

Varying the problem

- Why is √3 irrational?

- Suppose $3m^2 = n^2$
- Now n^2 is divisible by 3 …
- … so n is divisible by 3 …
- … so $n = 3s$ and $3s^2 = m^2$ …
- … so m^2 and m are divisible by 3 …
- … so $m = 3r$ and $3r^2 = s^2$ …
- … and we have infinite descent again

Varying the problem

- Why is √4 irrational?

- Suppose $4m^2 = n^2$
- Now n^2 is divisible by 4 …
- … so n is divisible by 4 …
- … so $n = 4s$ and $4s^2 = m^2$ …
- … so m^2 and m are divisible by 4 …
- … so $m = 4r$ and $4r^2 = s^2$ …
- … and we have infinite descent again
- This is unfortunate, as √4 is rational

Varying the problem

- Why is $\sqrt{4}$ irrational?

- Which step is wrong?
- If m^2 is divisible by 4 , m is divisible by 4
- Why is this correct for 2 and 3 but not for 4?
- Is it because 2 and 3 are primes?
- Is it correct for 6?
- Is it correct for 12?

Asking questions

- Can you deduce that $\sqrt{6}$ is irrational from the fact that $\sqrt{6} = \sqrt{2} \times \sqrt{3}$?
- Can you deduce that $\sqrt{12}$ is irrational from the fact that $\sqrt{3}$ is irrational?
- Is it enough to prove that $\sqrt{p}$ is irrational for any prime to show that $\sqrt{N}$ is irrational when N is not a perfect square?
- Can you deduce that $\alpha\beta$ is irrational from the fact that α and β are irrational?

Asking questions

- Can you deduce that $\sqrt{2}+\sqrt{3}$ is irrational from the fact that $\sqrt{2}$ is irrational?
- Can you deduce that $\alpha+\beta$ is irrational from the fact that α and β are irrational?
- Can you deduce that $\alpha+r$ is irrational from the fact that α is irrational and r is rational?
- There are many challenging questions you can pose without using specific proofs of irrationality

Tackling the problem

- Why is $\sqrt{2}$ irrational?

- Let us consider another approach to the original problem.
- The previous questions have suggested that we might look at prime factorisations.
- We have $2m^2 = n^2$
- Does anything stand out about the prime factorisations of these two terms?
- Focus on the factor 2 – what can be said?
- It has an odd index in $2m^2$ and an even index in m^2
- Hence these two quantities cannot be equal

Varying the problem

- Why is $\sqrt{2}$ irrational?

- Does this method work for $\sqrt{3}$?
- Does it ‘fail’ for $\sqrt{4}$?
- Can we make it work for $\sqrt{6}$?
- Can we make it work for $\sqrt{12}$?
- More homework!
- It turns out that this is a resilient method

Tackling the problem

- Why is $\sqrt{2}$ irrational?

- Let us look at this another way
- It is a bit more sophisticated
- Suppose $\sqrt{2} = \frac{n}{m}$
- Now consider $\sqrt{2} - 1$
- This is positive …
- … and it is equal to $\frac{n-m}{m}$
- Hence we know that $\sqrt{2} - 1 \geq \frac{1}{m}$

Tackling the problem

- Why is $\sqrt{2}$ irrational?

- $\sqrt{2} - 1 \geq \frac{1}{m}$
- Now think about $\left(\sqrt{2} - 1\right)^2$
- This is positive and equal to $3 - 2\sqrt{2}$
- Hence it is $\frac{a}{m}$ for some positive a …
- … so we have $\left(\sqrt{2} - 1\right)^2 \geq \frac{1}{m}$

Tackling the problem

- Why is $\sqrt{2}$ irrational?

- $\sqrt{2} - 1 \geq \frac{1}{m}$ and $\left(\sqrt{2} - 1\right)^2 \geq \frac{1}{m}$
- This works for any power $\left(\sqrt{2} - 1\right)^N$
- We always have $\left(\sqrt{2} - 1\right)^N \geq \frac{1}{m}$
- But $\sqrt{2} - 1 < 1 \ldots$
- … and this is impossible

Tackling the problem

- Why is $\sqrt{3}$ irrational?

- Does the same method work?
- Suppose $\sqrt{3} = \frac{n}{m}$
- We always have $\left(\sqrt{3} - 1\right)^N \geq \frac{1}{m}$
- But $\sqrt{3} - 1 < 1 \ldots$
- … and this is impossible

Tackling the problem

- Why is $\sqrt{4}$ irrational?

- Why does this method fail to work?
- Suppose $\sqrt{4} = \frac{n}{m}$
- We always have $\left(\sqrt{4} - 1\right)^N \geq \frac{1}{m}$
- But $\sqrt{4} - 1 = 1 \ldots$
- … and this is perfectly sensible

Tackling the problem

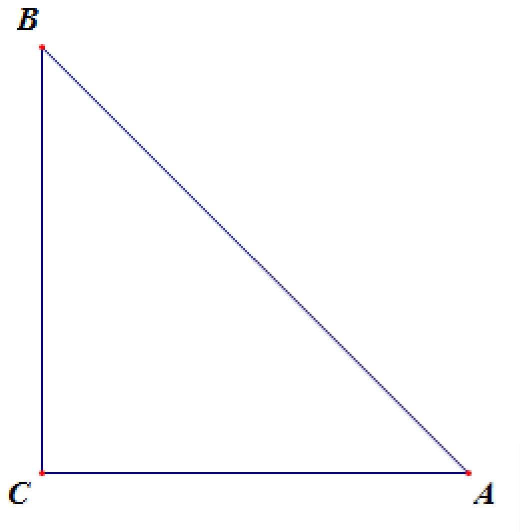
- Why is $\sqrt{5}$ irrational?

- How do we adapt this method?
- Suppose $\sqrt{5} = \frac{n}{m}$
- We always have $\left(\sqrt{5} - 2\right)^N \geq \frac{1}{m}$
- But $\sqrt{5} - 2 < 1 \ldots$
- … and this is impossible
- Again, this is a resilient method

Tackling the problem

- Why is √2 irrational?

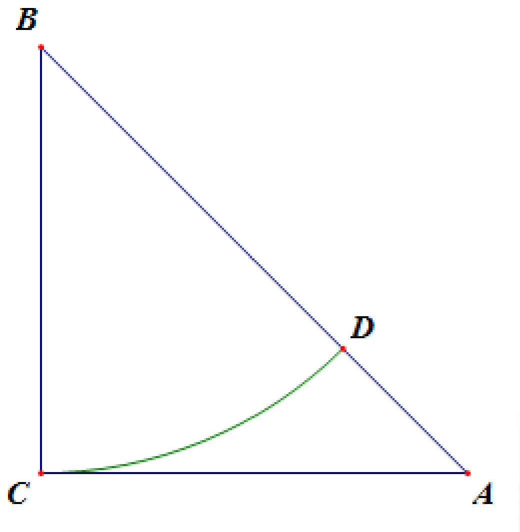
- We now take a geometrical approach
- Is there an isosceles right-angled triangle with integer sides?
- Suppose there is such a triangle



Tackling the problem

- Why is $\sqrt{2}$ irrational?

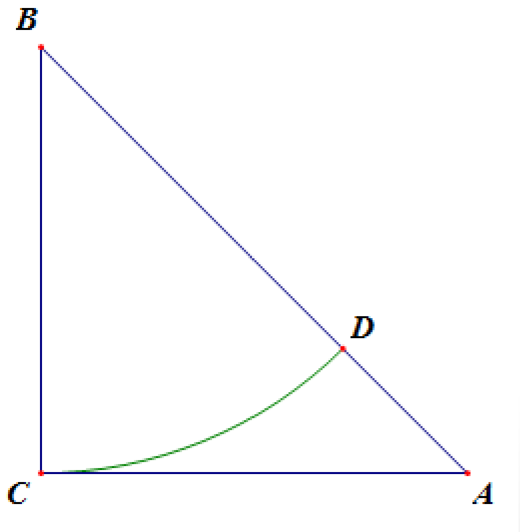
- We now take a geometrical approach
- Is there an isosceles right-angled triangle with integer sides?
- Draw an arc as shown



Tackling the problem

- Why is √2 irrational?

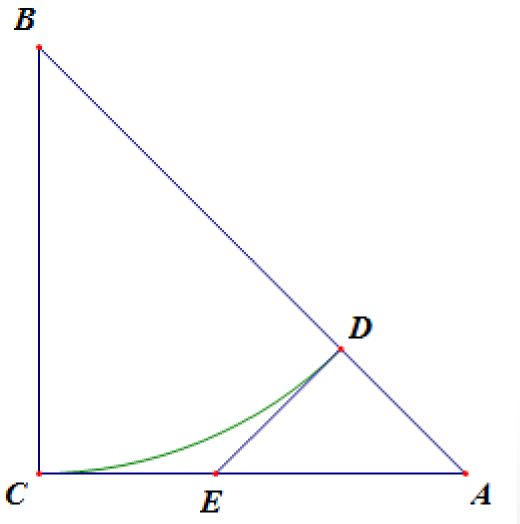
- We now take a geometrical approach
- AB and BC are integers…
- … and BD = BC…
- … so AD is an integer



Tackling the problem

- Why is $\sqrt{2}$ irrational?

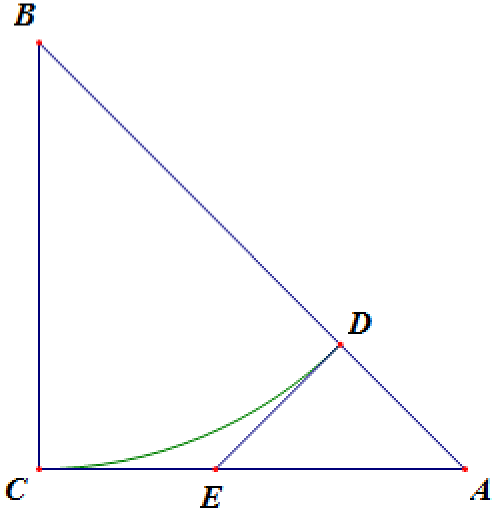
- Now draw DE perpendicular to AB as shown
- Now CE = ED = AD …
- …which is an integer
- So AE is an integer …
- … and ADE is an isosceles right triangle with integer sides



Tackling the problem

- Why is √2 irrational?

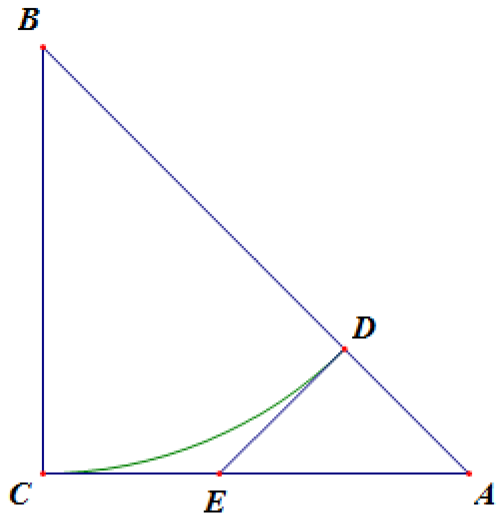
- ADE is an isosceles right triangle with integer sides
- And it is smaller than ABC…
- ... so we can do this again…
- … infinitely often …
- … only we can't …
- Why not?



Tackling the problem

- Why is √2 irrational?

- So there is no isosceles right triangle with integer sides
- But if √2 were rational …
- … there would be one …
- … so √2 is irrational



Tackling the problem

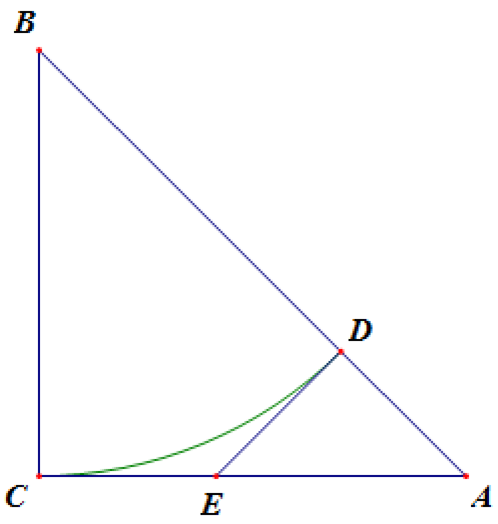
- Why is $\sqrt{2}$ irrational?

- If you like this proof, there are several variants using A4 paper folding in Nick Lord's Note in the March issue of *The Mathematical Gazette*
- We can make this argument algebraic, and then see if it can be generalised

Tackling the problem

- Why is √2 irrational?

- Let's look at the algebra …
- … a bit more sophisticated
- Let AC = m and AB = n
- Then AD = $n - m$ …
- … and AE = $2m - n$



- So $\frac{n}{m} = \frac{2m-n}{n-m}$

Tackling the problem

- Why is $\sqrt{2}$ irrational?

- The identity $\frac{n}{m} = \frac{2m-n}{n-m}$ is useful

- You can multiply this out to check it
- Now check that $0 < 2m - n < n$ …
- … and that $0 < n - m < m$ …
- … and you have a new algebraic argument by infinite descent

Varying the problem

- Why is √3 irrational?

- What is the useful identity now?
- It is $\frac{n}{m} = \frac{3m-n}{n-m}$
- We still have $0 < 3m - n < n \ldots$
- … and $0 < n - m < m \ldots$
- … so the proof works

Varying the problem

- Why is $\sqrt{4}$ irrational?

- What goes wrong?
- We still have $\frac{n}{m} = \frac{4m-n}{n-m}$
- But it is not true that $0 < 4m - n < n$ …
- or that $0 < n - m < m$ …
- … since $n - m = m$
- This cannot be ‘mended’, which is reassuring

Varying the problem

- Why is $\sqrt{5}$ irrational?

- What is the useful identity now?
- It is true that $\frac{n}{m} = \frac{5m-n}{n-m}$.
- But $5m - n > n$ and $n - m > m$ …
- … so the proof does not work
- Can we mend it?

Varying the problem

- Why is √5 irrational?

- How do we mend the proof?
- Try $\frac{n}{m} = \frac{5m-2n}{n-2m}$, which is true
- Now $0 < 5m - 2n < n$ …
- … and $0 < n - 2m < m$ …
- … and the proof works

Varying the problem

- Why is $\sqrt{13}$ irrational?

- What is the rearrangement now?
- It is $\frac{n}{m} = \frac{13m-3n}{n-3m}$, and it works
- So what is the general form?
- Homework!

Tackling the problem

- Why is $\sqrt{2}$ irrational?

- Let S be the set of integers n for which $n\sqrt{2}$ is an integer
- If S is not the empty set, it has a least element
- Let this be k
- Consider the number $(\sqrt{2}-1)k$
- Show that $(\sqrt{2}-1)k \in S$

Tackling the problem

- Why is $\sqrt{2}$ irrational?

- Let S be the set of integers n for which $n\sqrt{2}$ is an integer. Its least element is k
- We have $\left(\sqrt{2}-1\right)k\sqrt{2} = 2k - k\sqrt{2}$
- But $k\sqrt{2}$ is an integer, so this is an integer
- Hence $\left(\sqrt{2}-1\right)k \in S$

Tackling the problem

- Why is $\sqrt{2}$ irrational?

- Let S be the set of integers n for which $n\sqrt{2}$ is an integer. Its least element is k
- We have shown that $(\sqrt{2}-1)k \in S$
- However, $(\sqrt{2}-1)k < k$
- But this is a contradiction, since k was the least element of S

Tackling the problem

- Why is √2 irrational?

- Here is an approach described by Bob Burn in the latest *Mathematics in School*
- Start as usual with $2m^2 = n^2$
- Write this as $m^2 + m^2 = n^2$
- Clearly m cannot be even, since we would have infinite descent
- So m is odd

Tackling the problem

- Why is √2 irrational?

- Here is an approach described by Bob Burn in the latest *Mathematics in School*
- Write this as $m^2 + m^2 = n^2$
- So m is odd
- Now look at residues mod 4
- Squares are either 0 or 1 mod 4

Tackling the problem

- Why is $\sqrt{2}$ irrational?

- Here is an approach described by Bob Burn in the latest *Mathematics in School*
- Write this as $m^2 + m^2 = n^2$
- So m is odd
- Hence $m^2 \equiv 1 \pmod 4$
- So $n^2 \equiv 2 \pmod 4$ and this is impossible

Tackling the problem

- Why is √3 irrational?

- Write this as $m^2 + m^2 + m^2 = n^2$.
- Again, if m is even, so is n and we have infinite descent
- Hence m is odd, so $m \equiv 1 \pmod 4$ then $n \equiv 3 \pmod 4$ which is impossible

Tackling the problem

- Why is $\sqrt{4}$ irrational?

- Write this as $m^2 + m^2 + m^2 + m^2 = n^2$.
- Again, if m is even, so is n and we have infinite descent
- Hence m is odd, so $m \equiv 1 \pmod 4$ then $n \equiv 0 \pmod 4$ which is not a problem
- So now the method does not work, which is good

Tackling the problem

- Why is $\sqrt{n}$ irrational?

- This method is not 'global' since some values of n are difficult
- You should check that 5 can be done by choosing another modulus
- You should check that 7, 12 and 13 work by reduction to smaller cases
- However, 17 is much harder

A sophisticated approach

- Why is $\sqrt[k]{2}$ irrational (for $k > 2$)?

- As usual we have $2m^k = n^k$
- Hence $m^k + m^k = n^k$
- Now we appeal to a celebrated result proved by Andrew Wiles …
- … and this equation has no integer solutions
- Is this valid?
- Or is just a rather feeble joke?
- Time to stop

- One of my favourite artists, Paul Klee, once describing painting as going for a walk with a line.

- I sometimes think mathematics (and teaching it) is going for a walk with a problem.
- I hope you’ve enjoyed our little walk.