DECOMPOSITION
AND ALL THAT
ROT

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How do People Calculate?
Well, it varies rather a lot. Here are just three stories:

Me: “Do you know what 7 lots of 8 are?”
Peter (7): “No.”
Me: “Could you work it out?”
Peter: “I knew 10 8’s so I took away 8, that’s 72, and another, and another — 56.”

Me: “What’s 213 take away 188?”
Ray (adult): “25.”
Me: “How d’you do it?”
Ray: “Well it’s 12 up to 200 and 13’s 25.”

Me: “What’s 213 take away 188?”
Student (19): Silence
Me: “What happened when I asked?”
Student: “Panic.”
Me: “Did you see anything in your mind . . .”
Student: “Yes. I saw 213 written down and then 188 underneath, and then a line . . .”

Ask people how they calculate, and indeed observe yourself at work, and you will soon find a fascinating variety of idiosyncratic methods.

How are Calculations Taught?
In contrast to the diversity of methods people actually use, there is very little variation in the methods taught in schools. At the present moment, it is a fair bet that almost all children in this country are taught standard written algorithms for the four rules of number. By this I mean processes for addition, subtraction, long and short multiplication and division which are laid out something like this:

There are minor variations, of course, concerning whether you multiply first by the units or by the tens in long multiplication, and whether the quotient goes above or below the dividend in short division, and there is the relatively major distinction between decomposition and equal additions. But that’s about the sum of it. The vast majority of children are taught these methods as the primary way of dealing with whole number (and decimal) calculations; and for the majority, I suspect, these are the only methods they are taught.

The Nature of Standard Written Algorithms
Why should this be? The reasons lie largely in the nature of these algorithms. It is worth attempting to summarise them.

1. They are written, so the calculation is permanent and correctible.
2. They are standardised: it is possible to arrange that everyone does the same thing.
3. They are contracted in the sense that they summarise several lines of equations involving distributivity and associativity.
4. They are efficient. For instance, it’s less efficient to add the tens first, because you might need to make a subsequent amendment to their total after you’ve added the units.
5. They can be automatic: they can be taught to, and carried out by, someone who has no understanding of what is happening.
6. They are symbolic. One does one’s calculations entirely by symbol manipulation, with no reference to the real world, or any other model. At the last stage the answer appears with, usually, no previous approach to it.
7. They are general in the sense that they will work for any numbers, large or small, whole or decimal, with few or many digits. This is perhaps their greatest attraction, and it comes from their exploitation of place-value.
8. They are analytic. They require the numbers to be broken up, into tens and units digits, and the digits dealt with separately.
9. They are not easily internalised. They do not correspond to the ways in which people tend to think about numbers.
10. They encourage cognitive passivity⁷ or suspended understanding. One is unlikely to exercise any choice over method and while the calculation is being carried out one does not think much about why one does it in that way.

Possibly the most significant of these characteristics is the eighth. By breaking a number up into hundreds, tens and units digits, and dealing with these as digits, we develop methods which can be applied to all numbers, however large or small, and which can be applied efficiently. However, such an analytical approach detaches the methods from the area of complete numbers (i.e. numbers not split into digits) where people are more at home. Thus even if the rules can be remembered they are learnt largely without reasons and are not related to other number knowledge. They are far from aiding the understanding of numbers; rather they encourage a belief that mathematics is essentially arbitrary.

As has often been stated, training in these methods may be a good idea if you want to produce clerks, and others, who are quick and accurate at doing large numbers of difficult calculations by hand. Also, teaching these methods leaves you with work which is easy to manage and to mark.

Perhaps a further point should be added to the list above: they are traditional. For a lot of non-specialist teachers of mathematics, as for the general public, the four rules of number are the standard written algorithms. Concept and algorithm are equated. So to teach division you teach a method rather than an idea.

As has been demonstrated in research directed to quite other ends, D. A. Jones² investigated the methods used by each of 80 11 year olds to calculate 67 + 38, 83 – 26, 17 x 6 and 83 - 79, 83 – 51, 83 – 7? of these, three were standard written algorithms.

The Use of Standard Written Algorithms

One of the most remarkable things about these methods is that they are used so little. In some research directed to quite other ends, D. A. Jones² investigated the methods used by each of 80 11 year olds to calculate 67 + 38, 83 – 26, 17 x 6 and 116 + 4². The questions were written in this form, and the children were free to use written or mental methods. Over half of the 320 calculations were successfully completed by non-standard methods, e.g. 83 – 26: 83, 73, 63, 60, 67; 17 x 6: 12 x 6 = 72, 72 + 30 = 102. Thus despite the heavy teaching of standard algorithms, they are not necessarily chosen for calculations of this order of difficulty. This suggests, at the least, that the standard methods are not suitable for mental work. Casual enquiry amongst both adults and children into how they do their “sums” yields similar results.

At the same time, the standard algorithms are not understood by children. Any teacher of maths to children between 8 and 16 will recognise these:

\[
\begin{array}{cccc}
3 & 5 & + & 4 \\
\hline
9 & 1 & 1 & =
\end{array}
\]

and probably quickly add to the sad collection. The situation is highlighted if the child who has failed on paper is asked the question orally and is able, as so often, to do it in his head. Mary Harris writes about the two different spaces in which the child operates: the space of written numbers and the space of spoken numbers⁸. She conjectures that there is only the most tenuous connection between the two.

The standard algorithms are also mis-used, like sledge-hammers to crack nuts. How many children have recorded calculations like these?

\[
\begin{array}{cccccc}
1 & 0 & 0 & 0 & \times & 1 0 0 \\
9 & 9 & 5 & \hline
0 & 0 & 0 & 0
\end{array}
\]

Either being required to conform, or not expecting to think for themselves, they apply standard methods as though they were always the most appropriate.

The Nature of Mental Algorithms

Work these in your head, or better still ask a child, and try to determine how they were done:

\[
\begin{array}{cccc}
5 7 & + & 2 4 & = 8 3 & - & 6 9 & = 3 & \times & 2 4 & = 1 1 2 & + & 4 \\
\end{array}
\]

Here are some of the characteristics of mental algorithms. They contrast with those in the list above, but there has been no attempt to match them point for point.

1. They are fleeting and often difficult to catch hold of.
2. They are variable. From his 80 children, Jones² recorded 16 different methods altogether for finding 83 – 26. Of these, three were standard written algorithms.
3. They are flexible, and can be adapted to suit the numbers involved. Do you have different methods for 83 – 79, 83 – 51, 83 – 7? 4. They are active methods in the sense that the user makes a definite, if not always very conscious, choice of method and is in control of his own calculations.
5. They are usually holistic, in that they work with complete numbers rather than separated tens and units digits, e.g. 4 x 35 = 2 x 70 = 140; 4 x 28: 4 x 30 = 120, \(-8\), 112.
6. They are frequently constructive, working from one part of the question towards the answer, e.g. 37 + 28: 37, 47, 57, 67, 65.
7. They are not designed for recording. So written down they tend to sprawl, as in the example above. But they can of course be recorded where this is desirable.
8. They require understanding all along. A child who gets his mental calculations right almost certainly understands what he is doing. Equally, their use develops understanding. But on the other hand they cannot be used to achieve performance in advance of understanding.
9. They are often iconic. Either they relate to an icon such as the number-line or a number-square, or they depend upon serial enunciation as in 32 + 21: 32, 42, 52, 53. In either case some overall picture of the numbers is being used.
10. They often give an early approximation to the correct answer. This is usually because a left-most digit is calculated first, but in the context of complete numbers, e.g. 145 + 37: 175, 182; 34 x 4 = 120, 136.
11. They are limited in the sense that they cannot be applied to the most difficult calculations, such as 269 x 23. Nevertheless they are suitable for a greater
range of problems than a casual observation of school number work might lead one to suppose.

It is fairly clear that mental methods are the ones to foster if you wish to use and develop children’s understanding of number, and any teaching of them would obviously be accompanied by other means to this end. You would have to accept that children would not be able to do calculations before they had a pretty clear idea of what was going on. You would have to be able to provide them with alternative ways of dealing with difficult calculations. And a teacher of these methods would obviously expect a degree of independence and individuality in his pupils.

So teachers, or schools, or society, have a choice in the methods which might be taught, and the choice can be made on the basis of previously determined aims in number education. Either standard written algorithms for efficiency and order and because that is what we have taught for 100 years, or mental algorithms for independence and understanding and because they are the methods people actually use. Or both: “Ah, mental algorithms are all very well, but they must learn the standard methods sooner or later.”

Must they?

A Spectrum of Calculations

In order to discuss what we should teach, it helps to take a look at the range of calculations we can do with numbers. It is easy to include “The four rules of number” as a basic skill for all citizens without considering exactly what this entails. The general applicability of the standard algorithms makes us forget the enormous width of the spectrum (see diagram). For convenience, I have divided the range into five rough bands. They are approximately in order of decreasing frequency of use, and I suspect the relationship is a negative exponential one. (When did you last need to do a long multiplication, other than in your professional work?) Needless to say the order is also one of increasing difficulty. This is very convenient: the calculations we use a lot are the easy ones. It should also give us occasion to pause before spending a lot of energy on teaching processes which will be used very little. See Moore and Williams for figures on frequency of use of some calculations.

<table>
<thead>
<tr>
<th>Red</th>
<th>Orange</th>
<th>Yellow</th>
<th>Green</th>
<th>Blue</th>
</tr>
</thead>
<tbody>
<tr>
<td>5+9</td>
<td>135+100</td>
<td>139+28</td>
<td>592+276</td>
<td>3964+7123+4918+5960</td>
</tr>
<tr>
<td>13−8</td>
<td>85−20</td>
<td>83−26</td>
<td>592−276</td>
<td></td>
</tr>
<tr>
<td>4×7</td>
<td>5×30</td>
<td>17×3</td>
<td>931×8</td>
<td>931×768</td>
</tr>
<tr>
<td>35+6</td>
<td>90+3</td>
<td>72+4</td>
<td>693+7</td>
<td>8391+57</td>
</tr>
</tbody>
</table>

I have put some typical calculations in each band, but have no wish to make very precise distinctions. In general terms the descriptions of the bands are these:

**Red band** This is the only one clearly defined and it contains number bonds up to 10+10 and 10×10 and their inverses. It is highly desirable to have all these facts available for instant recall.

**Orange band** These are roughly addition, subtraction, multiplication and division with a number with a single non-zero digit and for which everything is fairly straightforward. They can all be done by a one-step mental process, given thorough knowledge of results in the Red band. It is to these calculations that one most frequently sees standard written algorithms inappropriately applied.

**Yellow band** This covers the range of calculations for which mental methods are entirely appropriate. The average person in the street, given a practical, motivating context, would do these in his head. So can average 11 year olds, given encouragement. It is for calculations in this band that one so often sees the disjunction between a child’s space of written numbers and his space of spoken numbers.

**Green band** These could be done mentally but on the whole few people would want to, or need to.

**Blue band** In a practical situation it would be absurd to use a mental process for these. If a calculator were available, it would be equally absurd to use a written method.

Proposal

I think that the reasons for teaching the standard written algorithms are out of date, and that it is time we all took notice of this. I believe there is a place for mental algorithms, for the use of calculators, and for ad hoc, non-standard written methods. I think a large amount of time is at present wasted on attempts to teach and to learn the standard algorithms, and that the most common results are frustration, unhappiness and a deteriorating attitude to mathematics.

Therefore propose, for consideration and criticism, an alternative approach to the teaching of elementary arithmetic. Very roughly, children should go through three stages, progressing according to their ability.

Stage 1 The acquisition of mental techniques for calculations in the Red, Orange and Yellow bands.

Stage 2 The use of calculators for Green and Yellow bands.

Stage 3 The development of some casual written methods.

I must stress that I am referring to only one aspect of children’s mathematics. Calculations are only a part of number work, and number only a part of mathematics.

I want to elaborate a little on these three stages (particularly the first) but clearly I cannot do more than give an outline. I make no reference to children’s ages, since the age at which they can begin to understand a given aspect of number work is so variable.

Stage 1

I imagine there is little controversy in the suggestion that Red and Orange band calculations should be done in the head. Nevertheless it is important to remember that children do take quite a time to appreciate facts like 47+10=57 and 13×10=130. This is a measure of the subtlety and power of our place-value system. It is also an indication of the nature of children’s problems: if they are not certain of facts like these, they should not be rushed to calculations like 47+18 or 13×12.

Two sorts of understanding have to be fostered at this stage. First the understanding of numbers and the place-value system. One tool which I think is invaluable for this is the number-line. This version is commonly met in infants classes:

```
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
```

but for bigger number the place value structure needs to be emphasised:

```
0 10 20 30 40 50 60 70
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Further extensions to hundreds, thousands, fractional numbers and negative numbers are also invaluable. Use of these number-lines can help to give children a
personal, overall picture of the numbers (and how they relate to each other) which I think is often missing. Mathematicians seem commonly to possess an iconic number-line permanently available in their heads, and we might attempt to give children a similar icon.

The other part of understanding is of the meaning and nature of the four operations: of multiplication as repeated addition, of division both as repeated subtraction and as the opposite of multiplication, of subtraction both as “take away” and, perhaps more importantly, “and how many?”. This last point seems to be particularly important. Both large and small subtractions make more sense in terms of complementary addition: how does one work out 15 – 8, 311 – 275?

Given understanding of and facility with calculations in the Red and Orange bands, and some personal picture of the number system, children should have little difficulty in developing methods for dealing with Yellow band problems. The usual mental algorithms seem to be based on serial enumeration, as in 47 + 34: 47, 57, 67, 77, 81; or on successive approximation, as in 17 x 4: 20 x 4 = 80, 76, 72, 68; or on some iconic such as a number line: 110 – 88:

\[ \begin{array}{c|c}
80 & 90 & 100 & 110 & 120 \\
\hline
20 & 30 & 40 & 50 & 60
\end{array} \]

It is perfectly possible for these methods to be recorded, and doing so may help children to be aware of their mental operations. If free format recording is allowed, children are less likely to develop the all too common inhibition about expressing their own mathematical thoughts. As they become confident, they will abbreviate what they write. The aim is an internalised process which is understood, rather than an externalised one which may not be.

**Stage 2**

Understanding of larger numbers is a necessary preliminary to using them in Green and Blue band calculations. Given this, and the understanding of mental methods for the previous bands, children will be able, and likely, to make sensible use of calculators for more difficult problems. Their personal understanding of number should make it unlikely that they will ask “Is it an add, Miss?” and press the wrong button. They will be able to use Orange band calculations to check more difficult problems. Their personal understanding of larger numbers is a necessary preliminary to using them in Green and Blue band calculations. A person who has to do a lot of such calculations will soon develop methods of a brevity suited to his needs. Here is a non-standard method for long multiplication:

\[ \begin{array}{c|c|c|c|c}
& 20 & 3 & 10 & 200 & 30 \\
& 8 & 160 & 24 & 414 & \\
\hline
32.80 & 42.80 & 52.80 & 54.80 & 54.75 & 21.95
\end{array} \]

which shows how much diagrams can help. The important thing is a process which is intelligible (to the user), rather than one which is standardised or quickest.

**A Great Opportunity**

The advent of calculators has provided us with a great opportunity. We are freed from the necessity to provide every citizen with methods for dealing with calculations of indefinite complexity. So we can abandon the standard written algorithms, of general applicability and limited intelligibility, in favour of methods more suited to the minds and purposes of the users. Up to now I don’t think this point has been sufficiently appreciated. In his stimulating, and otherwise excellent article, Michael Girling writes: “The most refined methods of long division, for instance, . . . need not be taught.” 6 Why teach any “refined” methods? They would only be needed by people who do a lot of calculations, and for such people a calculator is a better aid than pencil and paper.

A difficulty in advocating a programme such as this is the common confusion of concept with algorithm. The argument starts from the quite acceptable premise that children should learn how to calculate and arrives at the conclusion that they should be taught the standard algorithms. Calculators can then be seen as a great bogey in schools. The emphasis must be shifted to the other end of the argument. Children should be helped to acquire sensible methods for calculating, and for the majority of calculations met in everyday use these will be mental methods. Teaching mental techniques will not lead to children doing less calculations in school: probably they will do more. More importantly children will acquire a better understanding of number from using their own mental algorithms than from the repeated application of standard algorithms they do not comprehend. With mental methods occupying their proper place as the principal means for doing simple calculations, the position of calculators is clear. They are the sensible tool for difficult calculations, the ideal complement to mental arithmetic.

**References**