"IS IT AN 'ADD', MISS?"

Part I

by Margaret Brown and Dietmar Küchemann.

The Concepts in Secondary Mathematics and Science (CSMS) project, financed by the SSRC, was set up at Chelsea College in October 1974, with the aim of investigating the levels of conceptual demand of various mathematical and scientific areas within the secondary curriculum. Having charted the parts of these subjects children find difficult, and tried to explain why, we hope eventually to be in a much better position to match topics and teaching approaches to the styles of thinking of individual children.

In the area of mathematics, we decided to start almost at the beginning and look at the understanding of the basic four operations on natural numbers among children mainly in the first year in secondary school, although looking briefly at children one year above and below this age.

We were interested not so much in computation as in whether children could see what operation should be applied in a given situation, and conversely whether given an arithmetic expression like $9 \times 3$ they could make up a "story" for which the expression was a model. (This second area is the subject of Part II of this article.) We wrote a number of items like the three shown in Fig. 1, of which some involved small numbers (less than 10) and others larger ones (between 12 and 500).

We tried to vary the form of an operation involved; for instance the shop item above is more akin to the "Cartesian product" or "cross-product" form of multiplication (see Computation and Structure 3, §4, of the Nuffield Mathematics Project) than to the "repeated addition" form of multiplication which was embodied in the remaining two multiplication items in the test. Again the chocolate item above involves grouping (how many shares if they contain 3 each?) whereas the daffodil item is of the "sharing" type (how many each if they are shared between 23?)

We used these items to interview 35 children from all ability levels in the first and second year of the secondary school. The children were asked how they would tackle the problem, sometimes going as far as carrying out the steps if their strategy was not clear, and were only then asked to choose an expression from the sets shown on the right in Figure 1, which were on the reverse side of the problem cards.

Results of the Interviews

We classified the replies to the "problem" items into five different levels.

Stage 1: Intuitive. These children sometimes gave delightful idiosyncratic answers which showed they were playing the game on their own terms rather than on ours. Others, not knowing where to start, "guessed" wildly.

Daffodils

YG You er... I know what to do but I can't say it...
MB Yes, well you do it then. Can you do it?
YG Those are daffodils and these are flowerbeds, large you see... Oh! they're being planted in different flowerbeds, you'd have to put them in groups...
MB Yes, how many would you have in each group? What
would you do with 23 and 391, if you had to find out?
YG See if I had them. I'd count them up. say I had 20 of each ... I'd put 20 in that one, 20 in that one.
MB Suppose you had some left over at the end when you've got to 23 flower beds?
YG I'd plant them in a pot(!!)

**Daffodils**
DPI About 200
MB About 200. How did you work it out?
DPI Just guessed.
MB Just guessed. O.K. which of those did you do?
DPI No.
MB No. O.K.

**Stage 2A: Early Concrete.** Children were able to produce successful strategies for most, if not all, the items but these were always based on addition or subtraction.

**Daffodils**
MB (Reads question.) What would you do with those two numbers to work it out?
SB 23 there take away 23, 23, 23 ... MB Keep on taking away 23?
SB Yes.
MB Can you think of any of those that would be right?
(Back)
SB That one.
MB 23 take away 391. Any others?
SB That one.
MB 391 take away 23.

**Stage 2A/B: Middle Concrete.** Multiplication and division were sometimes recognised, but there were some inconsistencies. The sandwich item in particular was likely to cause trouble.

**Chocolate**
MBa (Reads question to himself) 4?
DK Mmm ... how did you work that out?
MBa 3 times 4.
DK 3 times 4 is what?
MBa 12.
DK I see, how did you realise it was 4?
MBa Just worked it out in tables.
DK I see O.K. Which of these? (Back).
MBa 3 4's.
DK Would any of the others do?
MBa Yes, 12 shared between 4 ...
DK Why would that one work?
MBa Cos it's the same ... that works out as 12 ... so does that, so it's the same.
DK Well ... MBa 4 shared between 12 works out 3 and so does that ...
DK I see, what about this one though, 12 divided by 3?
MBa No ... it only goes 3 times ... that's 3, 6, 9, 12, that's right innit? No that's 4! Yeh that one works as well.
DK O.K. So we've got three versions here now, let's just sort of pin it down a bit more ... if we look at what we started with; 12 squares altogether, 3 squares in a row, O.K. that's what you're told, which of those three would you think the better one?
MBa 3 times 4.

**Stage 2B: Late Concrete.** Fairly clear identification of multiplication and division items, although often the inverse was chosen, e.g. $3 \times 4$ for the chocolate item. No distinction between $391 \div 23$ and $23 \div 391$ was made.

**Chocolate**
MB (Reads question).
JN 4.
MB What did you do with the 12 and the 3.
JN Divided it.
MB (Back).
JN 3 divided by 12 and 12 divided by 3.
MB Are they the same?
JN Yes.

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**Figure 2: Question Facilities**

<table>
<thead>
<tr>
<th>Chocolate</th>
<th>Daffodils</th>
<th>Books</th>
<th>Shoes</th>
<th>Cakes</th>
<th>Ducks</th>
<th>Pay Off</th>
</tr>
</thead>
<tbody>
<tr>
<td>MBa 4</td>
<td>DK 3</td>
<td>SB</td>
<td>MBa</td>
<td>MBa</td>
<td>MBa</td>
<td>MBa</td>
</tr>
<tr>
<td>MBa 3</td>
<td>DK 12</td>
<td>MBa</td>
<td>MBa</td>
<td>MBa</td>
<td>MBa</td>
<td>MBa</td>
</tr>
</tbody>
</table>

**Stage 3A: Early Formal.** Generally very quick answers to all questions, and a recognition of the non-commutativity of subtraction and division.

**Daffodils**
LD How many 23's go into 391.
BJ Which one of those would ...?
LD This one.
BJ That's right, now what does that one say?
LD 391 divided by 23.
BJ Can we use that one? (23 \div 391).
LD No, because that can't go into that.
BJ That's right.

One general point that we noticed throughout the interviewing was that almost all children could produce successful strategies for solving problems, even when they did not recognise the operations involved. Complementary addition (subtraction by "adding on" in the "shopkeeper's method") was successfully used mentally with large numbers by many children who could not recall the official subtraction algorithm. Also various, sometimes ingenious, forms of repeated addition and subtraction were employed by many who, although recognising that the problem required a multiplication or division operation, confessed that they could not remember the "rules".

There would therefore seem to be an argument for considering whether the traditional algorithms are the most useful ones and whether a method more closely related to children's "concrete" methods might not be easier to recall, even if slightly less efficient.

After interviewing we then tried out various versions of group tests which, although they clearly gave less information on any individual child than the interviews, did enable a whole class to be tested at once. Since the very straightforward items on addition and those on the "take away" form of subtraction were done correctly by almost all children, we omitted these and arrived at a final format which included, in Part 1, three problem items on harder forms of addition and subtraction, three problems on multiplication and three on division. (In Part 2 were the "story" items on subtraction, multiplication and division which will be discussed in the second part of this article.) In each part there was a trial item for discussion, which included revision of the operation symbols, and the teachers were also asked to take classes through the test together, reading all the items out twice, to avoid the difficulties with poor readers.

Results were gathered on 1,138 children in the first...
Results on the items
The item facilities, i.e. the percentages of first year secondary children who got each question correct, were as shown in Fig. 2. (The item was marked correct provided a child recognised a correct strategy, so that for example for the chocolate item the answers 12 + 3, 3 + 12 and 3 × 4 were all accepted.)

The items which are linked are those which were parallel in form except that one contained small numbers (indicated by small dots) and the other large. We judged that these differences, which are usually around 10%, were due in each case to those children whose grasp of the operation was rather tentative, so that large numbers pushed it beyond their threshold.

Comparison between the operations was difficult since some items represented very straightforward forms (e.g. buckets, cakes, daffodils) whereas all the others were chosen to represent more complex forms. Also whereas there was little difference between the two forms of division, with multiplication the "Cartesian product" item (shop) was significantly harder than the "repeated addition" form. In the story items which will form the subject of Part II the order was very clearly +, −, × in increasing order of difficulty; the results of Part I although more confused do not contradict this.

We tried to match the performance of children in the class test to the levels on the interviews which have been described above, and although the correspondence is rather tenuous we arrived at the table below. (The percentages are likely to be correct only to within about 4% for the 11-12 sample, within 7% for the 12-13 sample, and within 9% for the 10-11 sample.)

<table>
<thead>
<tr>
<th>Stage</th>
<th>Intuitive</th>
<th>Concrete Operations</th>
<th>Formal Operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>10-11</td>
<td>2%</td>
<td>23%</td>
<td>38%</td>
</tr>
<tr>
<td></td>
<td>(7%)</td>
<td>(14%)</td>
<td>(36%)</td>
</tr>
<tr>
<td>11-12</td>
<td>2%</td>
<td>15%</td>
<td>40%</td>
</tr>
<tr>
<td></td>
<td>(4%)</td>
<td>(8%)</td>
<td>(24%)</td>
</tr>
<tr>
<td>12-13</td>
<td>1%</td>
<td>12%</td>
<td>30%</td>
</tr>
<tr>
<td></td>
<td>(3%)</td>
<td>(6%)</td>
<td>(21%)</td>
</tr>
</tbody>
</table>

The percentage of children recognising the difference between 84 + 28 and 28 + 84 was rather more than those in the 3A category; the total at 10-11 for instance was around 18% while that at 11-12 was 20%, and at 12-13, 25%. However, not all these children had a firm grasp of multiplication.

The figures in brackets are the percentages estimated to be in the corresponding Piagetian stage by the science team of CSMS, using a class form of an original Piagetian task concerning volume and density, with 5,000 children. In the assessments of both our test and the science tasks it is particularly difficult to distinguish where the middle sub-stage of concrete operations begins and ends, and the two sets of figures more nearly correspond if the three substages of concrete operations are taken together.

In Part II of this article we will discuss the results in the "story items" together with some of the apparent reasons for the difficulty of the concept of multiplication.

MAT IN MATHS

This statement from members* of the 9-16 Sub-committee of the Teaching Committee of the Mathematical Association is a follow-up to the articles on Mathematics for the first two years in Secondary School by Miss R. F. Gibbons (Chairman), M. J. Cahill and W. E. Walker which appeared in the March 1976 issue of Mathematics in School.

The abolition of the 11+, the coming of comprehensive schools and the creation of middle schools, together with the changes in primary school methods, have led to children with a very wide range of mathematical abilities and very varied mathematical backgrounds coming together in a new school.

It has been recognised that the attempt to test the ability of these children on arrival turns out to be a test of the primary schools from which they have come. This situation has led more and more secondary schools to adopt mixed ability learning groups for mathematics at least in the first year, which has meant a reappraisal of methods. All teachers are affected by this, from the most experienced to those who have not yet finished their training.

What is being done to assist teachers in this adjustment to new situations? We offer the following suggestions for action — and questions to stimulate further discussion.

1. Training

The following action could be taken:

(a) Colleges and departments of education should include much more instruction in, and discussion of, mixed ability methods in their courses. How many of the tutors have practical experience of these methods?

(b) Support must be given to probationary teachers who may have received little or no information about such methods. Advisers and teachers’ centres must help with this. Are probationers released for such support and training?

(c) Provision needs to be made for the retraining of experienced teachers. Again, advisers, DES and teachers’ centres should be involved. What facilities exist for the release of teachers for retraining?

2. Dissemination of Information

There is a vast pool of experience in secondary, middle and primary schools in mixed ability methods yet enthusiasts are frequently working in isolation in many parts of the country.

(a) What opportunities are there in your area for cross fertilisation of ideas and information?

(b) Who is responsible in your area for the correlation of information on mixed ability groups?

(c) When did you last talk about teaching methods to your primary/secondary/college of education colleagues?

(d) Is there a possibility of an interchange of teachers between schools on a day basis in your area?

(e) Could some training college personnel gain experience in mixed ability group work in school, at the same time releasing teachers with experience to be made available to other schools in the area?

In the present financial difficulties, with goodwill, much can be achieved on the basis of these suggestions even without a large budget.

* See page 34.