SOLVING LINEAR ALGEBRAIC EQUATIONS

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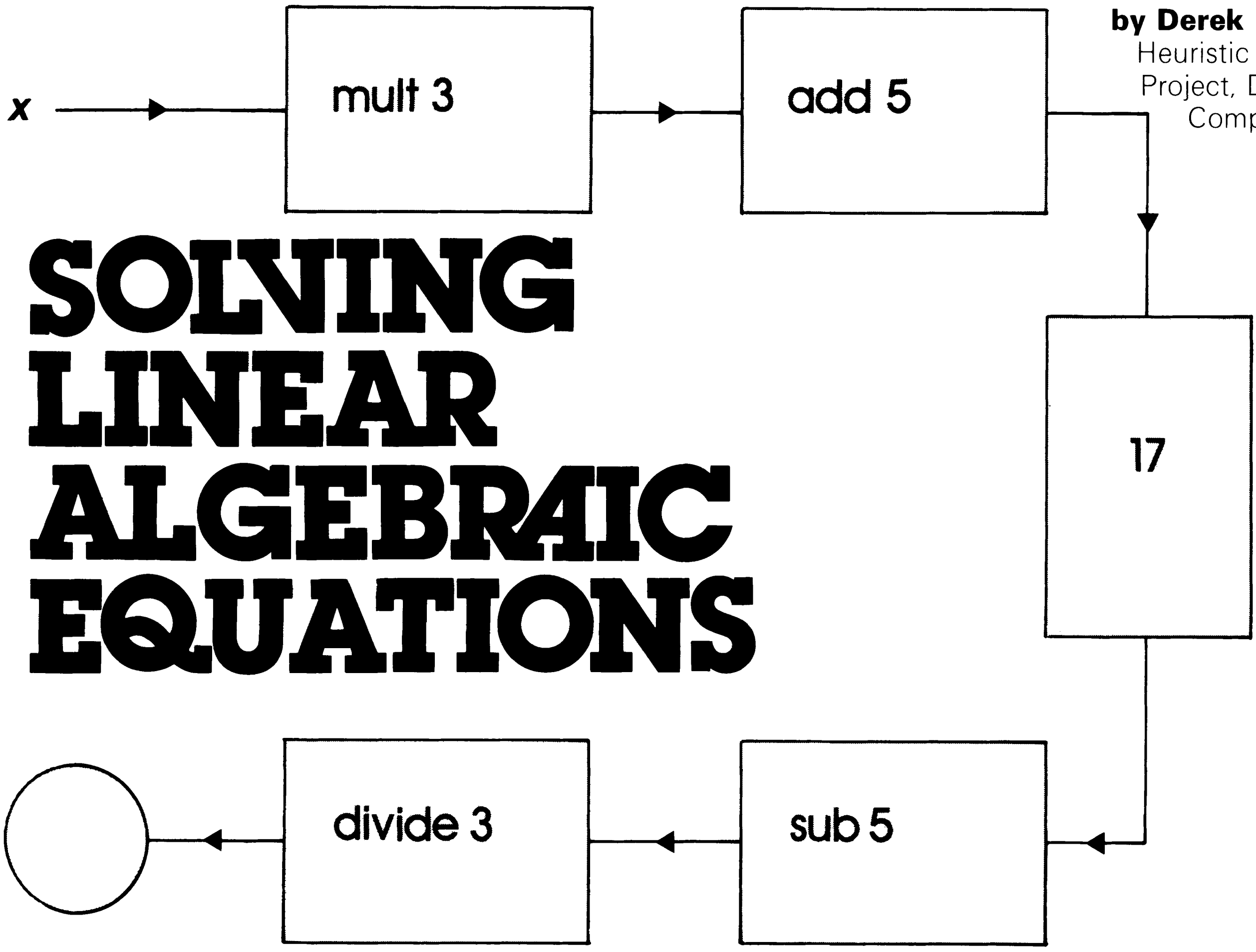


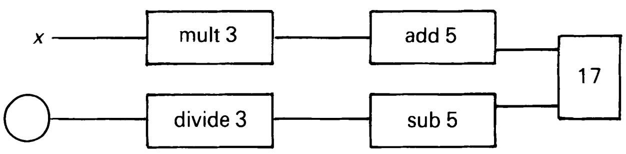
This paper reports the errors made by 60 fourteen year old students when they solved simple linear algebraic equations by the "flow chart" method. The work was carried out while the author was at the University of Leeds.

Introduction

There have been several studies recently to determine pupil's performance on basic algebraic tasks. The "Chelsea" study[1] presented the pupils with a range of tasks, and concluded that a high proportion of pupils had a variety of conceptual difficulties with this domain. Sleeman[2] has reported a detailed study which shows the sorts of errors which pupils make when they are taught to solve linear algebraic equations by a "rule based" method (e.g. expand brackets, collect all the x-terms on one side, all the numeric terms on the other side, simplify expressions, etc.). Currently a study of "horizontal" arithmetic is being undertaken and will be reported subsequently. Herscovics and Kieran[3] have reported the results of a study which attempts to present students with a sequence of examples which enables them to construct the meaning for the concept of an equation, and Booth[4] reports a diagnostic teaching programme designed to "prevent" the formation of many of the errors reported in earlier studies.

In this study we looked at the performance of a group of pupils who had been taught to solve algebraic equations by the algorithmic or flow chart method[5]. This method teaches the pupil to solve the task, $3x + 5 = 17$ as indicated in Figure 1. That is initially the pupil is required to extract the operations from the equations and to note the operations on the "top" line of the chart and on the "bottom" line to specify the reverse of these operations (see Figure 1a). The equation is then solved by performing the operations specified in the "bottom" line on the chart, the several operations being applied from right to left, as shown in Figure 1b. Note that using this approach this same flow chart should be produced for the equation, $5 + 3x = 17$. Figure 2 shows the complete flow-diagram for solving the task $(x + 2)*3 = 21$.

(a) Laying out a flow chart to solve the task $3x + 5 = 17$.



(b) Solving the several operations in the flowchart, working from right to left. The intermediary results are placed on the arcs, and the final result is placed in the circle.

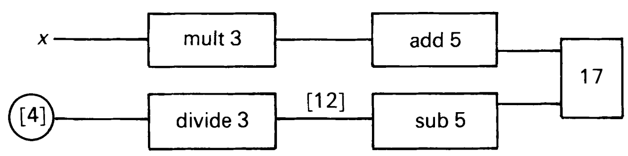


Fig. 1

Complete flow diagram for solving the task, $(x + 2)*3 = 21$.

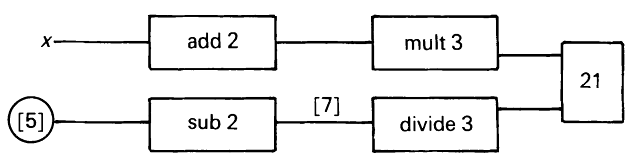


Fig. 2

In this experiment we asked 60 pupils to work a set of 17 tasks which included either 3 or 4 tasks of each of 5 types; below we list the types and an example of each:

(i) $mx = n + p$ $\quad 5x = 8 + 2$
(ii) $mx + nx = p$ $\quad 2x + 4x = 18$
(iii) $mx + n = p$ $\quad 2x + 3 = 9$
(iv) $m + nx = p$ $\quad 3 + 2x = 9$
(v) $mx = nx + p$ $\quad 4x = 2x + 6$

where m, n and p are integers.

Essentially, we were only expecting the pupils to be able to solve tasks of type (i), (iii) and (iv) as (ii) and (v) (those including $2x$s) had not been covered in class at that point. These were included to complete the comparison with the other group and to see whether they could generalise the method learnt to cover these more "advanced" types. Essentially the analysis given here will be for tasks of type (i), (iii) and (iv), with merely a few passing comments about (ii) and (v).

Results

As noted above, a set of 17 tasks, were worked by 60 fourth form pupils, the average age being 14 years 9 months, in a Leeds Senior High School. These were the top and the middle maths streams of this year. The pupils had been introduced to this method a year or so before the experiment was conducted. After the paper-and-pencil exercises had been completed, these were marked by the investigator, who then proceeded to carry out in-depth interviews with selected pupils to verify the nature of their misunderstandings.† A summary of the results of this experiment is:

1. A sizeable number of pupils search for solutions in all the types of tasks, and thus were not using the flow chart method.
2. Most difficulties were encountered with tasks of type (iv).
3. A sizeable number of pupils were unable to solve tasks of the form $mx = n$, when $m > n$.
4. Various other types of errors occurred with lesser frequency.

We shall now look at these categories in some more detail:

Pupils who search for solutions

Some pupils clearly did not use a flow chart method to solve tasks, but frequently merely wrote down the answer. When asked about their "method" in the interview they generally explained that they first substituted $x = 1$ and evaluated both sides of the equation, if that did not balance they would then try $x = 2$, $x = 3$, etc. The detailed summary is:

11 pupils used the "search" method

- 7 pupils used it on all types of tasks, (i), (iii) and (iv).
- 4 pupils used it only for tasks of type (iv). These pupils used the flow chart method incorrectly for tasks of type (iv), BUT recorded the answer as given by the search method. (One presumes that these same students also used a search method on tasks of type (i) and (iii) and allow their flow charts to "stand" as they gave the same results as their informal methods). (See also next section.)

Errors on tasks of type (iv)

16 pupils in all had difficulties with these tasks. The detailed breakdown being:

- 10 pupils got all 3 tasks of this type consistently incorrect. One pupil got 2 wrong and two pupils got 1 wrong. Most interesting as noted above, of the 10 pupils who got all three tasks wrong, 4 of them who recorded the incorrect procedure — gave the *correct* solution to the tasks, which we assume was achieved by searching for the solution. All the pupils in this group processed the task from left to right and so wrote a flow diagram which essentially corresponds to the equation:

$$(x + m)^* n = p$$

- One pupil wrote a flow chart that corresponded to the equation:

$$(x + m)^* m = p$$

and did this consistently for all 3 tasks. (Thus for the equation $3 + 2x = 9$, he wrote the flow diagram which corresponded to $(x + 3)^*3 = 9$.)

- Another pupil solved all 3 equations as:

$$m + nx = p \Rightarrow, \ x = p/n - m‡$$

so $3 + 2x = 9 \Rightarrow x = 9/2 - 3$

- Another pupil solved a task as: $m + nx = p \Rightarrow x = p - m$

Whereas with the other two tasks he used the "standard" error noted above.

Tasks of the form $mx = n$

11 pupils were unable to solve tasks of this kind correctly when $m > n$. 7 left the task unworked; 4 returned the answer $x = n - m$, thus $6x = 8 - 3$ was worked as $x = -1$.

Other errors noted

- One student worked all three problems of the form $mx = n + p$ as $x = (n + p)^* m$, and so $5x = 8 + 2 \Rightarrow x = 50$.
- Several pupils incorrectly solved the final stage of the equation and inverted the answer. Thus sub-tasks of the form $mx = n$ were solved as $x = m/n$, and so $6x = 5 \Rightarrow x = 6/5$.

Conclusions

It is hoped that this analysis of errors will help teachers present this method more effectively. We have the following specific comments:

- To help eliminate the "search methods" introduce early on tasks which do not have small positive integers for solutions. Indeed presenting pupils with a series of tasks whose solutions are non-integers will rapidly bring this deficiency to the teacher's attention.
- Stress even more strongly, the need to *focus* on the x in tasks of type (iv). The error noted here came from the pupils processing the tasks in a "natural" way, namely from left to right.

In subsequent notes we will compare the types of errors which arise with this approach with errors noted with the "rule-based" method and speculate on the "causes". We believe that our current analysis of pupils errors with "horizontal" arithmetic will provide additional valuable insights about the nature of errors which occur with both methods of teaching the solution of algebraic equations.

†Remedial teaching was then carried out by a member of the school's maths staff for one member of *each* matched pairs. The matched pairs were constructed by the investigator after interviews; both pupils were considered to have comparible difficulties. The object of this exercise being to determine whether it is possible to correct pupil's errors in this domain once they have been identified.

‡"$\Rightarrow$" shorthand for "was solved as".

Acknowledgments

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