SYMmetryplus
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TREASURE HUNT – 16

Where is this bridge (hint: it is not in the UK) and who designed it? Can you see the parabola and its tangent?

Recent News

28,270 MILES - WAY TO GO!

In October last year a record was set for the longest single road route ever worked out.

The Travelling Salesperson Problem (TSP) was first posed in the 1930's by Merrill Flood who was looking to solve a school bus routing problem in the US and it asks the following question: *"Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city?"* This is an example of an optimisation problem and is of immense importance to businesses who need to make deliveries to many places as the shortest route between the warehouse and drop off points generally means considerable saving in time and money. The TSP also has applications in computing DNA sequences, aiming telescopes and designing computer chip circuits.

William Cook and a team of researchers at the University of Waterloo in Ontario, Canada calculated a round route that starts in Portland, Dorset and finishes in Weymouth, four miles away has at least 100 times more stops than the previous longest TSP. It links 24,727 hostelries in the UK.

EDITORIAL

Welcome back to a new term and a new year! We congratulate *Mathematical Pie* for their 200th edition appears this spring, the first issue being published in October 1950. Members of SYMS will receive their copy this term.

Congratulations also to Adam Mallis who won a £10 Amazon voucher for Treasure Hunts 14 and 15. In this edition of *SYMmetry*plus you will find another two pictures, the first one kindly provided by Frédéric Laurent. Please email your answer to these treasure hunt pictures to me at symmetryplus@m-a.org.uk by 31 March 2017 when a draw will take place for the winners.

In this issue we have three new contributors whom I am delighted to welcome: Po-Hung Liu who works at the National Chin-Yi University of Technology in Taiwan and uses mathematical fiction; Tom Roper who will be The Mathematical Association's President from April this year and Jai Sharma who encourages us to beat out some rhythms. Stan Dolan introduces a new topic for us: quadratic residues and Colin Foster compares scattergraphs. Andrew Palfreyman is working with large numbers and divisibility again while Mike Arnold provides some puzzles to keep us busy. Jenny Ramsden continues her historical writings with John Couch Adams and Graham Hoare writes about Adams' astronomical work as well as towering over us at Brighton. Paul Stephenson continues to write about topology with some mysterious creatures.

All the answers to the puzzles in this edition can be found on the *SYMmetry*plus page on the MA website: http://www.m-a.org.uk/symmetry-plus as well as an index to all the editions of *SYMmetry*plus. This edition of *SYMmetry*plus contains the 1000th article since it started: who will that be?

Articles on what inspires you are always welcome! Please send them to me at symmetryplus@m-a.org.uk

Peter Ransom

SQUARE NUMBERS

In *SYMmetryplus 61*, Autumn 2016, modular arithmetic was used to find a prime factor of $2^{83}-1$.

Modular arithmetic has many applications. Two of the greatest mathematicians of all time, Leonhard Euler and Carl Friedrich Gauss, were fascinated by the patterns made by square numbers when considered in modular arithmetics.

Johann Carl Friedrich Gauss

1777-1855

Leonhard Euler

1707-1783

One outcome of their work on this culminated in Gauss's "theorema aureum" or "golden theorem". Details of this major theorem can be found on the internet.

To start our investigation of squares and modular arithmetic, consider squares mod 12 i.e. on a normal clock!

n	n^2	n^2 mod 12
0	0	0
1	1	1
2	4	4
3	9	9
4	16	4
5	25	1
6	36	0
7	49	1
8	64	4
9	81	9
10	100	4
11	121	1
12	144	0

You might like to change the number 12 (called the base) and form some other tables of this type.

The box below proves the result mentioned in the next paragraph at the top of the next column.

A pair of numbers symmetrically arranged about the centre of a column correspond to $n = i$ and $n = b - i$, where b is the base. The corresponding values of n^2 are i^2 and $b^2 - 2bi + i^2$.

n	n^2	n^2 mod b
…	…	…
i	i^2	★
…	…	…
$b - i$	$b^2 - 2bi + i^2$	★
…	…	…

Then $b^2 - 2bi + i^2 = (b - 2i)b + i^2$.

Therefore $b^2 - 2bi + i^2$ differs from i^2 by a multiple of b.

In arithmetic modulo b we can write

$b^2 - 2bi + i^2 \equiv i^2$ mod b.

Therefore symmetrically positioned numbers in the n^2 mod b row are equal.

One of the many things you are likely to notice is that, whatever the base, the numbers in the right-hand column are always symmetrically arranged about the centre of that column. This is a result which we can prove with a little algebra (see bottom of last column).

You may have also noticed that the remainders when squares are divided by 12 are 0, 1, 4 and 9 and that these are themselves all squares. Although this may seem 'natural', very few bases other than 12 have this property! Indeed, you are likely to have found this yourself if you tried a few other bases.

For example, consider what happens when the base is 20. (Because of our 'symmetry' result we don't need to consider $n > 10$.)

n	n^2	n^2 mod 20
0	0	0
1	1	1
2	4	4
3	9	9
4	16	16
5	25	5
6	36	16
7	49	9
8	64	4
9	81	1
10	100	0

Here, one of the remainders is 5, which is not a square.

The remainders when squares are divided by a base b are called 'quadratic residues' modulo b.

CHALLENGE

- ***How many bases can you find which have the special property that all the quadratic residues are squares?***
- ***What else can you discover about quadratic residues?***

Stan Dolan

CRYPTOGRAPHY IN THE CHRONOS GAME: CREATIVITY IN MATHEMATICAL FICTION

Mathematical writing as a means for evaluating students' learning has received more attention in recent decades. Compared to taking paper-and-pencil tests or answering open-ended questions, writing mathematical fiction is seen as a more effective way of triggering students' imagination and creativity. The story introduced below is comes from a scenario of a mathematical fiction tilted "Chronos Game", written by Jia-Tai Li (李嘉泰), a student at National Central University in Taiwan.

The National Central University (NCU, Chinese: 國立中央大學, Kuo-Li Chung-yang Ta-hsüeh, or 中大, Chung-ta) was founded in 1915 with roots from 258 CE in mainland China.

"Chronos Game" is an adventurous story of four persons, Robert (an explorer), Owen (a medical doctor), Murray (a college student), and Allen (a violinist). According to ancient Greek mythology, Chronos is an old wise man in charge of time. Actually Robert, Owen, Murray, and Allen are avatars of the same person living in four different parallel-time spaces. The last chapter of "Chronos Game" describes how they eventually met together and were requested to defuse a space bomb by entering a passphrase in 30 minutes. If they couldn't make it, the four parallel-time spaces would collide with each other and break down. They were given a 12×12 grid table containing several English alphabets and a string of numbers 7,4,3,3,4,2,1,1,3,9,3,2.

According to ancient Greek mythology, Chronos is an old wise man in charge of time.

I	C	J	F	I	I	S	E	I	I	I	Q
O	A	A	I	L	S	E	L	I	L	Y	U
J	R	P	R	S	O	N	L	H	I	R	E
D	E	A	E	M	M	S	E	N	U	A	E
I	V	N	G	V	O	E	N	V	I	L	N
V	A	E	U	O	F	A	N	G	E	U	O
E	R	S	N	M	I	L	K	A	R	M	Y
R	Y	E	K	B	S	I	L	Y	S	I	A
V	I	I	N	V	I	I	I	I	X	N	T
I	M	I	I	K	O	T	A	K	V	I	A
L	P	R	F	I	N	I	K	E	I	U	K
K	S	F	E	D	I	R	E	C	T	M	E

The following conversations show their attempts to figure out the passphrase.

"It surely is a number puzzle." Allen said.

Murray added, "It should not be that simple. These numbers 7,4,3,3,4,2,1,1,3,9,3,2 do matter."

"The number probably ought to be regrouped and some numbers needed to be picked out." Owen said.

"'Let's just do it. No time to lose!" Robert urged.

(Five minutes later…)

"I, I, F, K, A, R cannot spell a word." Allen spoke after several tries.

"Could it be that we are not on the right track? If this is a Go game board, the top-left corner is the origin." Murray replied.

"It could be." Robert said, "Let's try again!"

(Another five minutes passed…)

"S, P, I, I, I, A, … It still does not make any sense." Allen broke the silence.

Owen seemed figuring something out and said, "I've obtained N, P, E, I, H, R by applying matrix calculation. They remind me of the word 'Susphrine', which is a drug for curing asthma. But I don't think the answer would be so weird."

"It is still not working." Murray was so depressed.

"Don't give up, or we all die. We must miss something." Robert encouraged everyone.

"Don't tell me the only thing we could do is to witness the four parallel-time spaces break down." Murray said sadly.

Right at this moment, Robert seemed thinking of something suddenly. He stepped toward the bomb and entered 'F, U, J, I'. The lights on the bomb stopped glittering. The bomb was defused!

"How did you do that, buddy!" Owen shouted excitingly.

"Ha, I was wondering why we were required to solve this puzzle. It appears unnecessary for us all to solve the puzzle. Actually, we are the clue!" Robert replied.

Robert continued, "What we need is our names, Robert, Owen, Murray, and Allen. The initial alphabets of our names make the word 'ROMA', leading me to notice the Roman numerals in the table. We had been misled to spell an English word."

"Then, how did you figure out FUJI?" Owen asked.

Robert added, "The table can be divided into 9 regions according to the Roman numerals form I to IX. If we go with the number 743-342-113-932 and assume that '743' denotes the region VII, the fourth row and the third column, then we get the alphabet 'F'. Following this rule, '342' refers to 'U', '113' is 'J', and '932' is 'I'."

"But how could you make sure your conjecture is correct? It might be that the second digit represents the order of row and the third digit means column?" Allen asked.

"Good question!" Robert replied, "Have you noticed that the arrangement of the alphabets in the four regions is not symmetrical except F, U, J, and I? So I was pretty sure my conjecture is correct. There is always only one solution!"

I	C	J	F	I	I	S	E	I	I	I	Q
O	A	A	I	L	S	E	L	I	L	Y	U
J	R	P	R	S	O	N	L	H	I	R	E
D	E	A	E	M	M	S	E	N	U	A	E
I	V	N	G	V	O	E	N	V	I	L	N
V	A	E	U	O	F	A	N	G	E	U	O
E	R	S	N	M	I	L	K	A	R	M	Y
R	Y	E	K	B	S	I	L	Y	S	I	A
V	I	I	N	V	I	I	I	I	X	N	T
I	M	I	I	K	O	T	A	K	V	I	A
L	P	R	F	I	N	I	K	E	I	U	K
K	S	F	E	D	I	R	E	C	T	M	E

Making up the above alphabet table requires not only basic knowledge in cryptography, but also imaginative creativity, such as the connection between the initial alphabets of their names ROMA and Roman numerals. Particularly, for avoiding ambiguity and making sure there is only one solution, which is the typical nature of mathematics, the author utilised the concept of symmetry. It indeed endorses the claim that writing mathematical fiction could be an effective way of developing students' imagination and creativity.

Po-Hung Liu

DSL HZMT GSRH HLMT?

RH GSRH GSV IVZO ORUV?

RH GSRH QFHG UZMGZHB?

XZFTSG RM Z OZMWHORWV

ML VHXZKV UILN IVZORGB

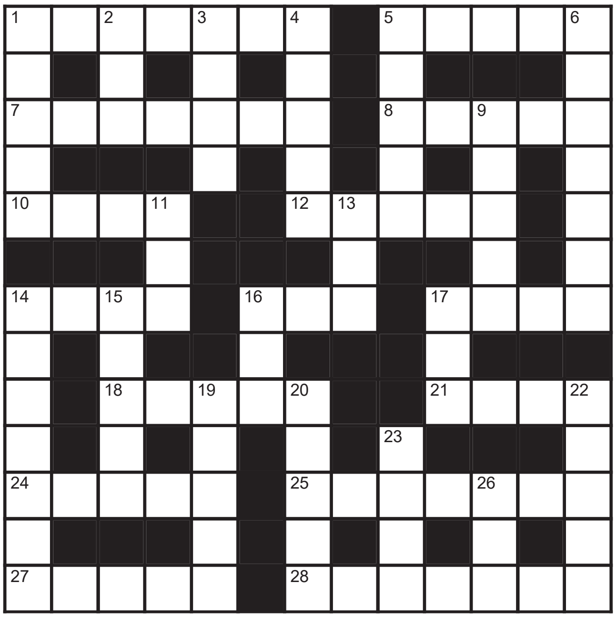
LKVM BLFI VBVH

OLLP FK GL GSV HPRVH ZMW HVV

R'N QFHG Z KLLI YLB, R MVVW ML HBNKZGSB

YVXZFHV R'N VZHB XLNV, VZHB TL

CROSSWORD



Across

1 Numbers based on ten (7)
5 The ___ of Hanoi puzzle (5)
7 A whole number (7)
8 Used for comparing (5)
10 Equivalent to three feet (4)
12 A probability outcome (5)
14 The edge of a shape (4)
16 Anagram of 'ten' (3)
17 In addition to (4)
18 Ten percent (5)
21 A quarter of a pint (UK or US) (4)
24 Ancient number system (5)
25 A type of distribution (7)
27 A compass direction (5)
28 Section of a circle (7)

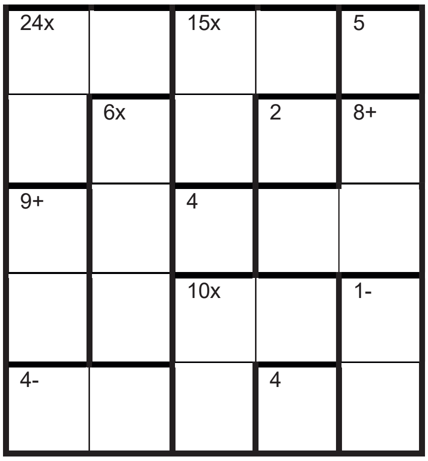
Down

1 Every day (5)
2 Hundredweight (abbreviation) (3)
3 One million (prefix) (4)
4 Big (5)
5 The second prime number (5)
6 A four-sided shape (7)
9 The entire amount (5)
11 Singular of 'dice' (3)
13 A British tax (3)
14 Three-dimensional circle (7)
15 Singular of 'data' (5)
16 A word found in logic (3)
17 Used with a geoboard (3)
19 0.11111 . . . as a simple fraction (5)
20 168 in a week (5)
22 Ultimate or maximum (5)
23 Like a doughnut (4)
26 'Money' less 'my' (3)

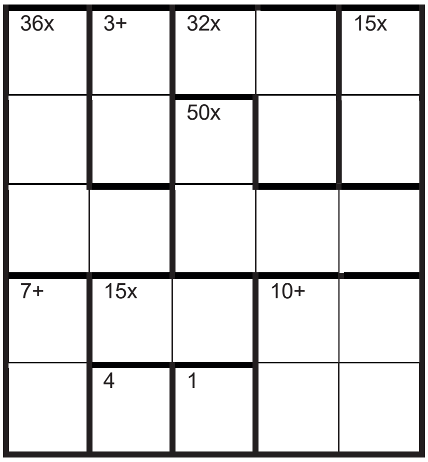
SYMKEN - 7

For an n by n grid all the digits from 1 to n must appear in every row and column. In each thick lined block, the target in the top is calculated from the digits in the block cells using the operation shown.

Easy



Moderate



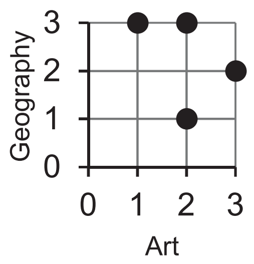
Mike Arnold

Perhaps this van with a Möbius band painted on it is really a Klein bottle! Notice that INTEGRAL is an anagram of TRIANGLE.

SCORES IN THREE SUBJECTS

Imagine that some students were asked to rate how much they liked three subjects: art, chemistry and geography. Each student gave a rating for each subject on a 1-3 Likert-type scale, where 1 is 'dislike' and 3 is 'like'.

Suppose that the graph for art and geography looked like this:

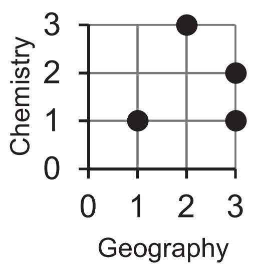


Can you tell how many students there were?

It is tempting to say 4, because there are 4 dots, but this might not be right. Can you see why?

A fifth student with the same pair of ratings as one of these four would give rise to a dot lying precisely on top of another dot, and so would not be discernible. So we cannot tell for sure how many students there were. All we can say is that there must have been at least 4.

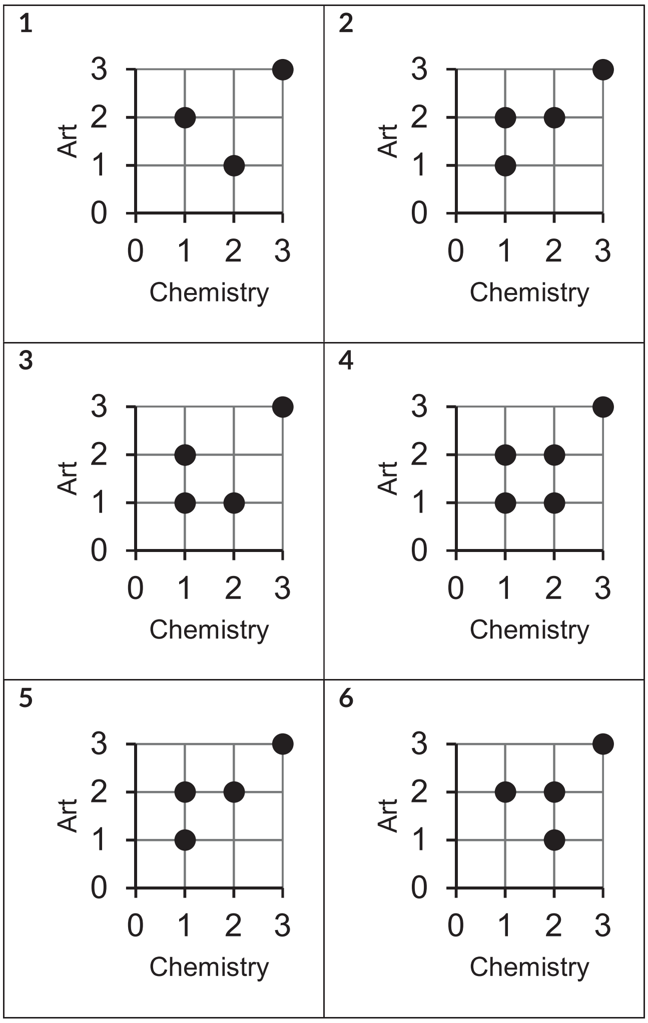
What if we now look at the geography and chemistry graph?



What can you say *for certain* now? Try to decide how much you can deduce at this point.

Before we look at the chemistry and art graph, what can you say about what it *must* look like, and what can't you be sure of without seeing it? Can you give an example of a graph that would be impossible? Can you give an example of a graph that *looks plausible* but is actually impossible?

Now look at these 6 chemistry and art graphs. Which one(s) could be correct and which cannot be correct, based on what you know from the first two graphs?



Actually, *all* of these 6 graphs are possible. (Are there any other possibilities?)

What can you deduce from each of these chemistry and art graphs?

What can you say about how many students there were?

Further questions you might like to think about are:

What happens with more students? Or more scale points? Or more different subjects?

What interesting examples can you concoct?

Colin Foster

THE 4 (MILLION) COLOUR MAP THEOREM

My subject is topology, 'rubber sheet' geometry.

This flat sheet of paper or a sphere are examples of a surface of *genus* 0. A doughnut or a coffee cup have genus 1. (Google 'Genus (mathematics)' and watch a coffee cup morph into a doughnut and back.) If you draw a circle on the first, you can always shrink it to nothing. But if you draw a circle round the top of a doughnut or the handle of a coffee cup, you can't. As you bore more holes through the doughnut or add more handles to the coffee cup, the genus goes up.

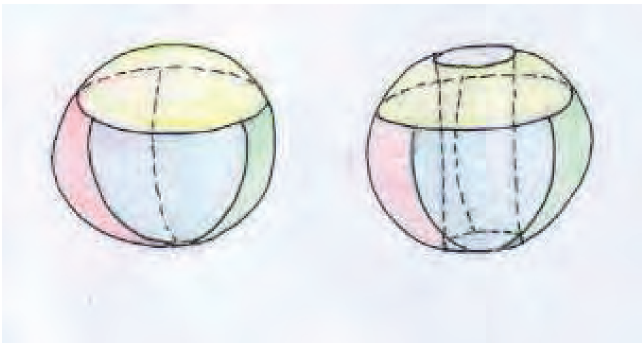
In simple terms, the value of a surface's genus is equal to the number of holes it has

When people mention 'The 4-colour map theorem', they're referring to a map drawn on a surface of genus 0. The fact that you never need more than 4 colours to colour a map, however complicated, has been proved, but not explained. You don't read one of the existing proofs and say, "Aha, *now* I see why the number must be 4". Before you attempt a proof yourself, be warned that the history of the problem is littered with ingenious, but ultimately flawed, attempts. (Go to nrich.maths.org/6291, 'The Four Colour Map Theorem'.) The intriguing thing about the '4' is that it is a *global* property of the map. I can run out of colours down in the bottom right-hand corner but can be sure that, even if I have to re-colour parts right out to the top left-hand corner, I can find a way to colour the map with my 4 colours.

The four colour theorem was proved in 1976 by Kenneth Appel and Wolfgang Haken. It was the first major theorem to be proved using a computer.

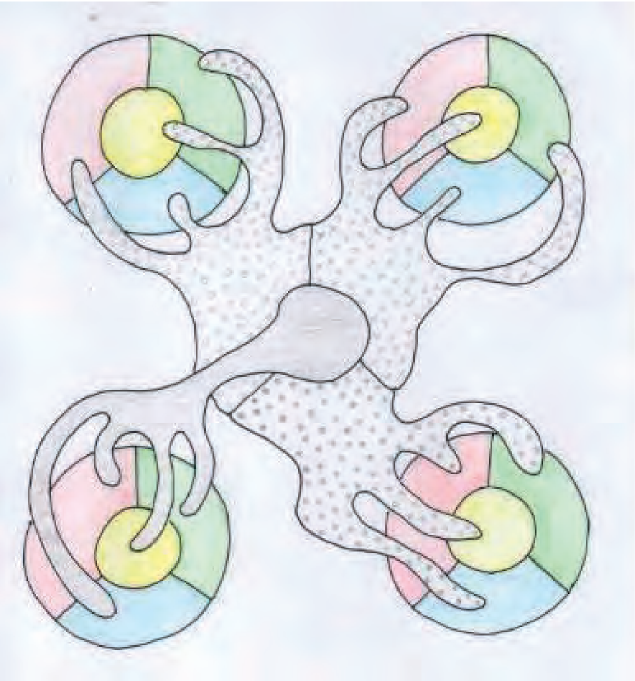
Before the four colour theorem, the five colour theorem stating that five colours suffice to colour a map was proved by P. J. Heaward in 1890.

Below, on the left, is a map on the sphere needing all 4 colours. On the right I have bored a hole through the sphere, thus increasing the genus from 0 to 1. The inside of the tunnel touches all 3 colours at the bottom and the colour at the top and therefore needs a 5th colour. It turns out that I can draw a map on this surface which may need as many as 7 colours. Go to nrich.maths.org/7027, 'Torus Patterns'. Join opposite edges of a square and you have a torus, so you don't have to find a model to draw maps on: you just draw them on the square. Click on 'Solution' and see the maps needing 2, 3, 4, 5, 6 & 7 colours sent in by a student. See if you can draw a map needing 7 colours which is simpler than Steve's.

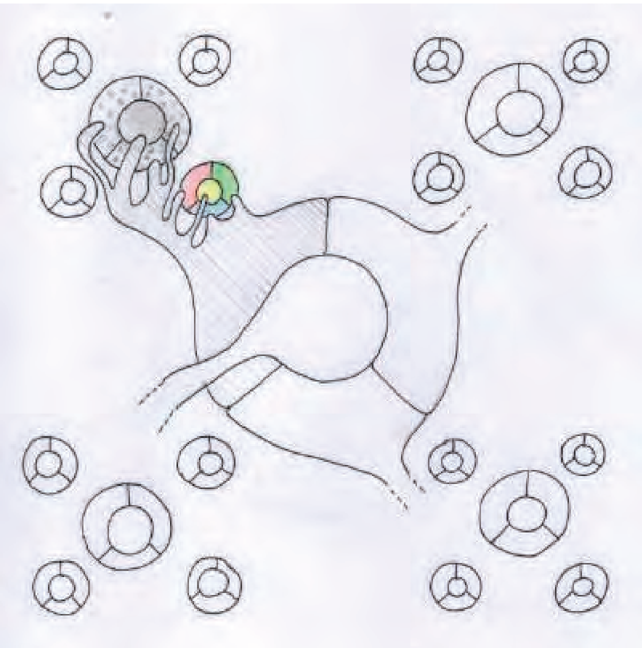


Can I increase the genus of my surface to make this so-called *chromatic* number as big as I want? Yes. Here's one way to do it.

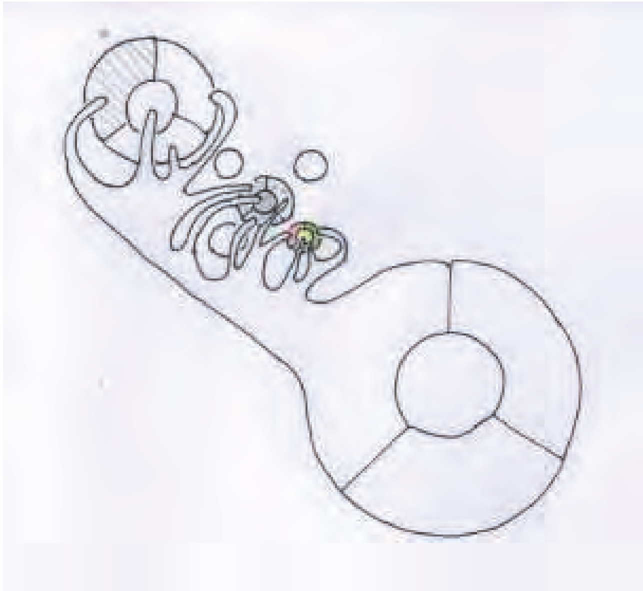
Top left in the first diagram opposite is our 4-colour map on a sphere. The surface of an animal (a 'tetrapus'?) forms one country of a new map. Since each tentacle touches one country on the old map, the tetrapus needs a 5th colour. But its head meets the heads of 3 similar tetrapuses to form a new sphere. So each tetrapus needs a different colour, a 5th, 6th, 7th and 8th. This is the first *iteration* in our operation.



On the next diagram the surface of a new animal (an octopus?) forms one country on a new map. A tentacle touches each country on both the old map and the one before. It therefore needs a 9^{th} colour. But its head meets the heads of 3 similar octopuses. So each octopus needs a different colour, a 9^{th}, 10^{th}, 11^{th} and 12^{th}. This is the second iteration in our operation.



For our third iteration we 4 have 'dodecapuses' like this, each touching 12 different countries, and needing a 13^{th}, 14^{th}, 15^{th} and 16^{th} colour:



We can continue like this, adding 16-puses, 20-puses, 24-puses, ..., 4*n*-pusses.

The Greek prefixes for 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 and 20 include duo-, tri-, tetra-, penta-, hexa-, septa-, octa-, ennea-, deca-, hendeca-, dodeca- and icosa- respectively.

What we have provided is a *constructive* proof (constructive because we've done it!) that surfaces can be devised on which maps can be drawn needing as many colours as we choose.

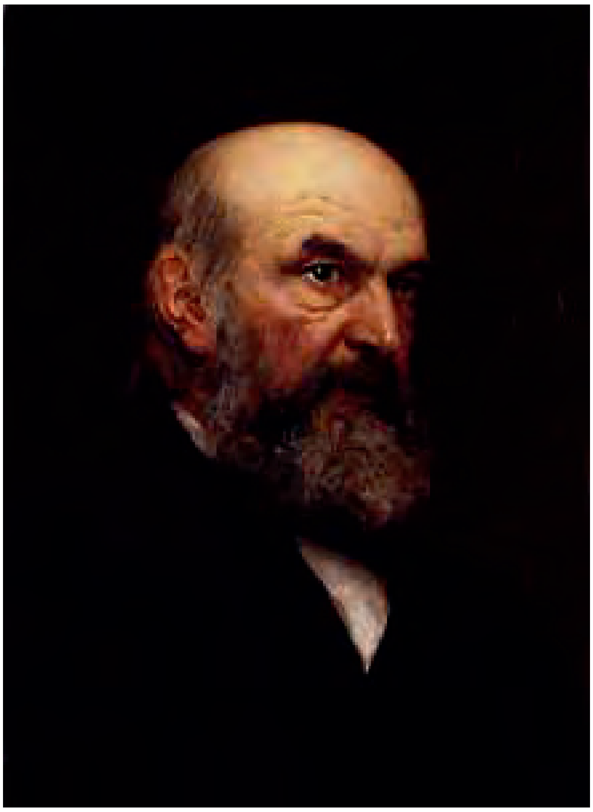
If you can find 16 colours, and a few friends, take an A1 sheet of paper and attempt a complete diagram for the third iteration. It will make a brilliant wall display.

There are 4 small spheres on our 8-colour map, 16 on our 12-colour map. How many will there be on your 16-colour map. How many would there be on a 4*n*-colour map?

Paul Stephenson

JOHN COUCH ADAMS

In the small church in Laneast, Cornwall, there is a black marble tablet commemorating the life of John Couch Adams, which records him as being the greatest mathematical astronomer that England has known since the time of Sir Isaac Newton. Adams is most famous for discovering the planet Neptune, but also remembered for his work on determining the motion of the Moon, and for ascertaining the period of orbit of the Leonids meteor shower.



John Couch Adams was born on 5 June 1819 at Lidcott farm near Laneast, some seven miles from Launceston in Cornwall. His father was Thomas Adams, a tenant farmer whose ancestors for at least four generations had been tenant farmers in or near Laneast. His mother was Tabitha Knill Grylls, also from a farming background. John was the eldest of seven children: Thomas Adams was to become a missionary in Tonga who completed the translation of the Bible into the Tongan language; George Adams carried on the farming tradition; William Grylls Adams was to become professor of natural philosophy at King's College London. Little is known of his three sisters: Grace Couch Adams, Mary Ann Adams and Elizabeth Adams, except that they pre-deceased John.

John read many books belonging to his mother who had inherited astronomy books amongst others.

John was named after his mother's uncle, John Couch, who had provided John's mother with some education and had left her the contents of his library, which included volumes on astronomy. Young John was a voracious reader and made great use of his mother's inherited books during his early education.

At the village school at nearby Laneast, John studied both the Classics and mathematics. A diligent student, he made swift progress and at age 12 his parents enrolled him at a private school at Devonport (which later moved to Saltash, then to Landulph in Cornwall) run by his mother's first cousin, the Rev John Couch Grylls. While at Devonport, John demonstrated a flair for mathematics and astronomy. He observed Halley's Comet in 1835 and calculated that an annular eclipse of the Sun would be visible in Lidcott in 1836, which he was able to observe. John devoured books on astronomy from his school's library and also from the library of the Devonport Mechanics' Institute, where he first encountered the works of Newton.

He could afford to go to university because his mother inherited some money and manganese was found on the farm.

The natural progression for John was to continue to university. Normally this would have been an expense beyond the means of a rural farming family, but the timely arrival of a small inheritance his mother received from an aunt, and the discovery on the farm of a source of manganese (an additive used in the steel-making process to harden metal) brought additional income to the family. John won a scholarship to St

John's College, University of Cambridge, entering the university in October 1839 to study mathematics as a "sizar" (a student who covered his tuition fees by tutoring other students). He graduated as Senior Wrangler (ranked top of the undergraduates in the final year examinations) in 1843, allegedly scoring double the number of marks of the second-placed student. Shortly afterwards he became a Fellow of St John's College.

George Airey's report about Uranus' irregular orbit intrigued John and he started to investigate it in 1843.

Whilst an undergraduate in 1841, Adams read the 1832 "Report on the Progress of Astronomy" by the Astronomer Royal, George Airy. This report mentioned the irregularities in the orbit of Uranus. Adams made a note in his diary that he had decided to investigate this problem as soon as he could after graduating, college duties permitting. True to his vow, he initially attempted a rough approximation in his graduation year, 1843. What Adams did not know at the time was that his successful solution of the problem would be shared by another mathematician, the Frenchman Urbain J Le Verrier, who was independently to discover the presence of the unknown planet almost simultaneously, but publishing his results in June 1846, and the planet subsequently identified in September that year. This was ahead of Adams, who published his calculations in January 1847. In his article "A Modest Cornishman", Graham Hoare explores in more detail the race to discover the unknown planet later known as Neptune.

Le Verrier gets the credit for discovering Neptune because he published his results seven months before Adams.

It transpired that Adams had indeed correctly predicted the likely position of Neptune ahead of Le Verrier, but the accolades for the discovery initially went to Le Verrier because he published his work first. Adams was very gracious on the topic, expressing his appreciation of Le Verrier's work and saying that he acknowledged Le Verrier's claim to the discovery. The two astronomers met at Professor Baden Powell's house in Oxford in 1847 (Baden Powell was Savilian professor of mathematics at Oxford at that time; he was father to Robert Baden-Powell, the founder of the Scout movement), and the meeting was entirely amicable.

When Queen Victoria visited Cambridge in 1847 for the installation of Prince Albert as Chancellor, she offered a knighthood to Adams, but he declined, either from modesty or perhaps fearing the financial liability that elevation to such a position might bring him. The following year, the University of Cambridge inaugurated the Adams Prize to commemorate his prediction of Neptune's position. The prize is still awarded every two years for the best essay in a mathematically-related subject.

After the discovery of Neptune, Adams turned his attention to considerations on terrestrial magnetism, on which he was to work periodically for the rest of his life. He was elected President of the Royal Astronomical Society in 1851 and in 1852 produced new tables for the Nautical Almanac that corrected serious errors from previous editions.

Adams had to leave St John's College in 1852, since at that time he could only continue his Fellowship there if he were to enter the clergy. He joined Pembroke College as a Fellow in 1853, and started work on the apparent acceleration of the Moon across the sky (caused by an extremely gradual reduction in Earth's rotation rate). This was to become Adam's second most important work of his lifetime after the discovery of Neptune.

In the autumn of 1858, Adams was appointed Professor of Mathematics at the University of St Andrews, where he stayed until May 1859. He missed Cambridge however, and after being made Lowndean

Professor of Astronomy and Geometry at Cambridge in 1858, he returned there, becoming Director of the Cambridge Observatory in 1861.

Adams met his prospective wife, Eliza Bruce (from Dublin), in October 1862 through mutual friends. They reacquainted their friendship two months later when Adams visited Dublin to ask Andrew Graham to become his senior assistant at the Cambridge Observatory. Adams proposed marriage to Eliza and they were wed on 2 May 1863.

Adams' third most important discovery was determining the orbit of the Leonids meteors. Professor H A Newton of Yale had studied historic records of thirteen sightings from AD 902 to 1833 and concluded that there were five possible orbits. Adams' investigations concluded that the disturbance of the larger of the planets to the orbit of the meteor shower agreed with just one of Professor Newton's orbits. Adams confirmed that the period of the orbit was 33.25 years, and correctly predicted in 1864 that the Leonid shower would next occur in November 1866 - coincidentally that meteor shower was particularly splendid. He subsequently refined his figure of 33.25 years by breaking up the meteor shower into smaller portions and considering their respective orbits. As before, with his work on Neptune, Adams did not publish many of his findings. In total Adams published around 50 papers, but his private papers show that he could have published many more.

In his later years, Adams worked on enumerating important constants, evaluating the next 31 Bernoulli numbers in addition to the 31 already known, and the Euler-Mascheroni constant, γ, (to 236 decimal places). He undertook with G G Stokes to catalogue Newton's scientific papers that Lord Portsmouth presented to the University of Cambridge in 1872, publishing an account of them in 1888. Adams also read widely, reading volumes on botany and history as well as fiction novels. It was said that he kept a novel close by him when working on long mathematical problems.

In 1881, Adams was offered the post of Astronomer Royal upon George Airy's retirement. Adams declined, citing advancing years and his inability to be as active as the role demanded. However, advancing years did not stop Adams from being active in mathematics almost to his dying day. In October 1889 Adams suddenly became severely ill, having suffered a stomach haemorrhage, but he recovered sufficiently to resume his normal mathematical work. His health faltered again in June 1890, when he suffered another similar attack, from which he never fully recovered. He died on 21st January 1892 at the Cambridge Observatory. He was interred at St Giles' Cemetery in Cambridge, now part of the Parish of the Ascension Burial Ground, near to the Cambridge Observatory. Three years after his death John Couch Adams was honoured with a memorial stone in Westminster Abbey. Eliza his wife survived Adams by several years, passing away in 1919 and was buried with him, their joint grave marked by a granite Celtic cross.

Jenny Ramsden

A MODEST CORNISHMAN

As a sixteen-year-old John Couch Adams observed Halley's comet at Landulph in his native Cornwall. Impressed by Halley's use of Newtonian mechanics to calculate the orbit of the comet his interest in astronomy was established. After graduation Adams set about investigating the irregularities in the motion of Uranus. The consensus among astronomers was that the perturbations of Uranus were caused by an unknown planet. The problem was to make assumptions about the interloper such as position, mass and orbit (distance from the sun and eccentricity). The myriad of possibilities was clearly formidable. Even so, his preliminary analyses strongly implied that there was, indeed, a planet *beyond* Uranus. By 1845, having combined the latest observations of Uranus with data from old records, Adams was confident enough to present his latest, refined, solution to James Challis, director of the Cambridge Observatory and to George Airy, then Astronomer Royal.

Adams was held up because Airey preferred collecting data to trusting mathematics.

But now came a period of frustration and disappointment for Adams. Although Airy, for example, had a strong mathematical background, he took the view that pointing telescopes at the behest of mathematicians was no substitute for collecting observations and taking measurements. Meanwhile, working independently, Urbain Le Verrier, a mathematical astronomer domiciled in Paris, was also trying to unravel the Uranus problem. While Airy and Challis dithered Le Verrier also faced indifference from the astronomical community in France and decided to seek help from Johann Galle, assistant to the Berlin Observatory. Galle, indeed, found the planet in September 1846 and wrote to Le Verrier that "The planet whose position you have pointed out actually exists".

Neptune's discovery by mathematical reasoning, rated a triumph for Newtonian mechanics, aroused great interest among the educated public. However, there was even more interest in an issue of priority as to which of the two, Adams or Le Verrier, deserved the credit for the discovery. Adams, self-effacing would, to his abiding credit, have none of it. Neither did he ever utter a word of complaint against Airy or Challis.

There was to be a twist in the tail of this episode of astronomical history. Now firmly located the next task was to calculate Neptune's orbit. Ironically, this exercise was simplified when it was realised, belatedly, that Challis had previously unwittingly observed Neptune three times but had not recognised it as the planet of Adams' prediction. The results of these deliberations were surprising for it transpired that the assumptions referred to earlier that Adams and Le Verrier had based their calculations upon were distinctly at variance with the actual ones. Fortuitously, the different methods they adopted had built in compensatory factors.

Adams made other contributions to astronomy especially to the complex problem of the Moon's motion. Beginning this work in 1851 he presented a paper to the Royal Society in 1853 in which he demonstrated a more accurate solution to the one given by Laplace who had omitted terms from his equations which were not negligible. His corrections to Laplace's work halved the discrepancy between the predicted and observed orbit of the Moon. The residual error we now know is attributable to the tidal drag of the Moon on the Earth which is slowing down the Earth's rotation.

Finally, we refer to another significant contribution Adams made to mathematics in the realm of the numerical analysis of differential equations (DEs). Students will be aware that indefinite integrals of such functions as e^{-x^2}, $\frac{e^x}{x}$ and $\frac{\sin x}{x}$ have no closed form solutions; that is they cannot be expressed in terms of known functions, (polynomials, sinusoidals, (sin, cos, …) and exponentials, (logarithms)). Similar situations arise with DEs and mathematicians, not to be outdone, have devised numerical methods of various kinds to 'solve' them. Adams and his collaborator, Bashforth, concentrated on first order DEs, for example

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy + 1.$$

The Adams–Bashforth methods were designed by John Couch Adams to solve a differential equation modelling capillary action due to Francis Bashforth. In 1883 Bashforth published his theory and Adams' numerical method.

Their procedure, the *Adams-Bashforth method*, is still effectively employed and is well-adapted to the modern computer.

Graham Hoare

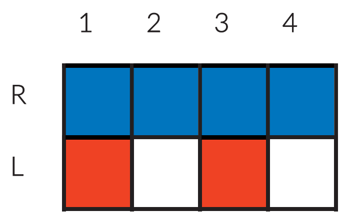
RHYTHM AND POLYRHYTHM

The word 'rhythm' is derived from the Greek word ῥυθμός (rhythmos) which means 'a regular recurring sound or motion'. Rhythms are everywhere; from the cycle of day and night to your own heartbeat. Here we will explore one of the most beautiful uses of rhythm: music.

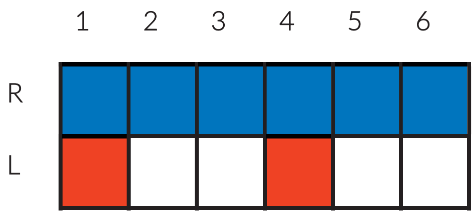
Rhythm

The diagram below shows a very simple example of musical rhythm. The top rhythm (R) consists of 4 evenly spaced beats and the bottom rhythm (L) consists of 2 evenly spaced beats. Since 4 is divisible by 2, both rhythms fit together in the same period of time. In fact, rhythm L is actually the same as rhythm R, except beats 2 and 4 are silent. The *phrase length* is the total number of beats it takes for the rhythm to repeat itself. This rhythm has a phrase length of 4 beats.

- Try playing the rhythms yourself. Use your right hand to play rhythm R and your left hand to play rhythm L.
- It may help to count as you play.
- Try playing the rhythms on different surfaces so you can clearly hear how they fit together.



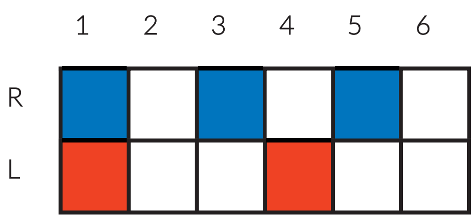
The next diagram shows another common musical rhythm. Rhythm R consists of 6 evenly spaced beats, and rhythm L consists of 2 evenly spaced beats. Since 6 is divisible by 2, they fit together in the same period of time and the phrase length is 6 beats.



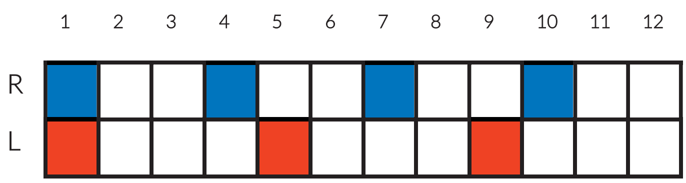
Polyrhythm

What happens when one rhythm is not divisible by the other, and they don't fit together in the same period of time? In this case we get what is called a *polyrhythm*, which means 'many rhythms'. Polyrhythms give a very exciting feeling to music and they are common in African music and music that derives from African music such as jazz, Brazilian Samba, and Afro-Cuban music. Polyrhythms are also used in Indian, Indonesian, and Western classical music as well as rock and dance music.

The diagram below shows a *3 against 2 polyrhythm*. Rhythm R consists of 3 evenly spaced beats, and rhythm L consists of 2 evenly spaced beats. Since 3 is not divisible by 2, both rhythms do not fit together in the same period of time, so the phrase length cannot be 3 beats. Instead, the phrase length must be 6 beats (some of which are silent) so that both rhythms can fit together in the same period of time.



The next diagram shows two a *4 against 3 polyrhythm*. Rhythm R consists of 4 evenly spaced beats, and rhythm L consists of 3 evenly spaced beats. In this polyrhythm, the phrase length is 12 beats.



Tasks

1. *What is the phrase length of a 5 against 4 polyrhythm?*
2. *Can you explain why?*
3. *Draw a diagram for a 5 against 4 polyrhythm.*

Jai Sharma

SYMdoku 2016-2017

Here is a SYMdoku puzzle. Every row, column and box of eight squares should contain the digits 0 to 7, not 1 to 8. Can you complete the puzzle?

2	**0**	**1**	**6**				
		3	5				
		2			4	3	
			4	1			0
6	3	0				4	
			1		2	0	3
	2		0				
	5		3	**2**	**0**	**1**	**7**

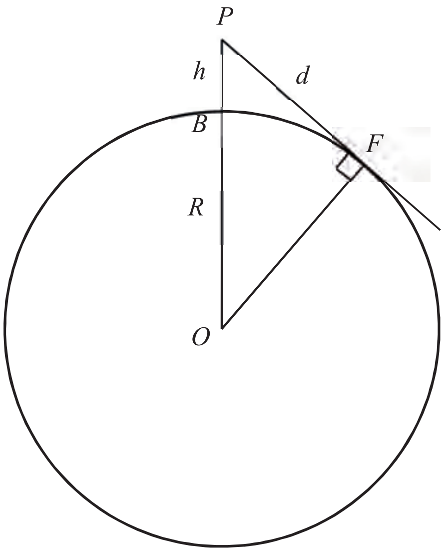
Mike Arnold

BRITISH AIRWAYS i360

In my youth I was warned not to believe everything I read in the newspapers, a warning that can be extended to other means of communication in our multi-media society.

Recently, a tower, the BAi360, of height 162 metres (531 feet) was opened on the seafront at Brighton. The passenger viewing pod, which rises from ground level to a height of 137 metres (450 feet), enables members of the public a 360° view of their surroundings. In the advertising details it is claimed that on a clear day the maximum range of sight is 26 miles. I decided to test this claim.

In the diagram below the picture, (not to scale), PB, which represents the tower from pod to ground, measures h feet, O is the centre of the Earth, radius R miles, and PF is a tangent drawn from P to the farthest point observable at F, where PF measures d miles.



By Pythagoras' theorem, bearing in mind that angle OFP measures one right angle, $d^2 = \left(\frac{h}{5280} + R\right)^2 - R^2$,

(recall, there are 5280 feet to a mile), so

$$d^2 = \frac{Rh}{2640} + \left(\frac{h}{5280}\right)^2.$$

The second term on the right side is very small compared to the other two, so we take, as a first approximation, $d^2 = \frac{Rh}{2640}$.

Take $R = 3960$, $h = 450$, then $d^2 = \frac{3960 \times 450}{2640}$. This is beautifully set up for cancellations! We have $d = 15\sqrt{3} = 25.98$ (to 2 decimal places). This is very close to 26. Even if we include the rejected term this is still the case. An impressive confirmation of the range of sight given in the blurb!

Graham Hoare

QUICKIE 32 NO CALCULATORS!

A certain number ends in 2. When the 2 is taken from the end of the number and placed at the beginning, a new number is formed which is 2 times the original. What is the original number?

FAREY SEQUENCES

I recently came across Farey sequences (or series) whilst researching phyllotaxis, the study of patterns exhibited by plants, and recalled that I had seen something about Farey sequences in *SYMmetry*plus. The Editor put me onto the article by Mary Partridge from Issue 43, Autumn 2010. Let me recap some of what Mary had to say.

A Farey sequence, $\mathcal{F}_n$, of order n is the sequence of all the irreducible fractions between 0 and 1, arranged in ascending order, whose denominators are less than or equal to n. 0 and 1 are included as $\frac{0}{1}$ and $\frac{1}{1}$ respectively. So, for example, $\mathcal{F}_6 = \left\{\frac{0}{1},\frac{1}{6},\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{2}{3},\frac{3}{4},\frac{4}{5},\frac{5}{6},\frac{1}{1}\right\}$.

Mary noted one particular property of Farey sequences and that is, if you have three successive terms, $\frac{a}{b},\frac{p}{q},\frac{c}{d}$, then the middle term, called the mediant, is given by, $\frac{p}{q}=\frac{a+c}{b+d}$.

Another property of Farey sequences is that if you have two successive terms, $\frac{a}{b},\frac{c}{d}$, then $ad-bc=\pm 1$. You can check both of these properties very quickly using the members of $\mathcal{F}_6$.

What is much less obvious is that for Farey sequences these two properties are equivalent, the one implies the other.

Two questions which intrigued me were, how did Farey sequences get their name? and how do you construct a Farey sequence for a given n? I found constructing even a small sequence quite tricky and wondered if there was an easy way to do this. The answer to the first question is interesting. John Farey (Senior) was a geologist and stated the first property noted above in 1816 in a note to the *Philosophical Magazine* but gave no proof. Cauchy saw the note and provided a proof in his *Exercices de mathématiques*. However, Hardy and Wright in their book, *An introduction to the theory of numbers*, claim that a mathematician called Haros stated and proved both properties in 1802, citing L.E. Dickson's, *History of the theory of numbers*. So it would appear that Farey can claim no credit for the statement of properties of Farey sequences or proofs of same! Yet still they bear his name and presumably will continue to do so.

In trying to construct the Farey sequence for $n=6$, I found that having to consider only the irreducible fractions complicated matters and for higher values of n getting the ascending order correct was tricky; not all fractions have easy or memorable decimal equivalents!! However, turning to Hardy and Wright for proofs of the properties, I found two potential methods for constructing a Farey sequence could be taken from the proofs. The first method shows how to construct the next term in a sequence given a particular term, and therefore much work to get the whole sequence. But the second is geometric, quick and easy, and works well for relatively small n .

The method depends upon the properties of what is termed a lattice. The array of squares formed by the integer points, i.e. points with coordinates (m,n) where m and n are integers, on a normal graph paper form a lattice. A lattice point P is said to be *visible* from O if, when we look along, the ray OP the first lattice point we see is P. Thus, in Figure 1 below, P is visible from O, but Q is not because the point R is seen first from O, looking along OQ, before the point Q.

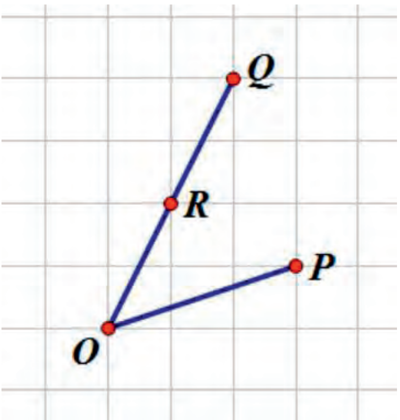


Figure 1

If we want to construct the Farey sequence $\mathcal{F}_6$ then we draw on graph paper or squared paper the following lines $y = 0$, $y = x$ and $x = n$, where in this case, $x = 6$. Now take a ray through O, initially along the x-axis, and rotate it anticlockwise so that it sweeps across the region defined by the drawn lines. As it rotates it passes through a sequence of visible points, the coordinates of which are (k, h) and each fraction $\frac{h}{k}$ is a member of $\mathcal{F}_6$ and they are generated in ascending order. See Figure 2 below.

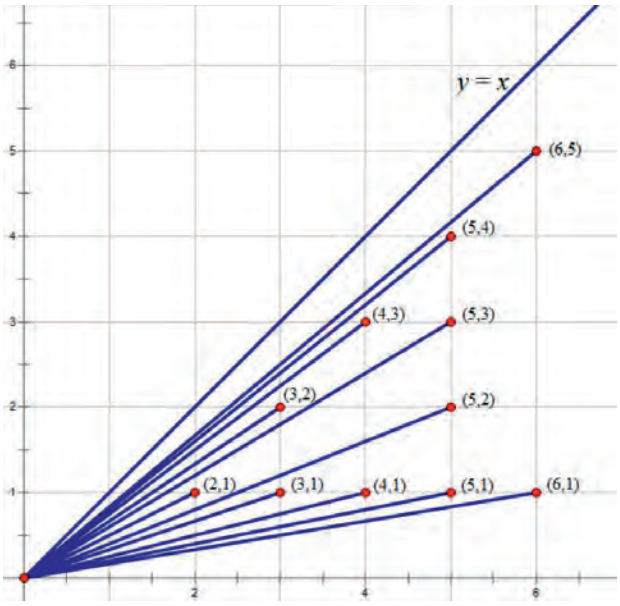


Figure 2

Tom Roper

SAUSAGE AND DOUGHNUT?

This surface of genus 1 was seen outside the cafeteria at Montpellier University. If you read Paul Stephenson's article on page 7 then he explains why this surface has genus 1. How many ordinary sausages do you think will fit in the sausage part? How many normal doughnuts would fit in the doughnut part?

THE COLLATZ CONJECTURE

There are many mathematical problems that look difficult, often owing to the presence of symbols or concepts that are unfamiliar. We have no chance of being able to solve these problems given that we do not even understand what they mean to start with! This is not the case with the Collatz Conjecture (which also goes by a number of other names).

The idea is very simple. You start with a positive integer. If the integer is even you halve it and if it is odd you multiply it by 3 and add 1. You then repeat the process. Here are three examples:

$5 \to 16 \to 8 \to 4 \to 2 \to 1.$

$11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1.$

$23 \to 70 \to 35 \to 106 \to 53 \to 160 \to 80 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1.$

The Collatz Conjecture is easy to state: the chain starting with any positive integer reaches 1 if you continue long enough. The problem is that Lothar Collatz proposed in it 1937 and still no-one knows how to solve it!

Lothar Collatz (July 6, 1910 – September 26, 1990) was a German mathematician, born in Arnsberg, Westphalia. In 1937 he proposed the Collatz Conjecture which remains unsolved.

You will notice from the chains above an overall unpredictability. The numbers in the chain go up and down forced by the simple rule generating the chain. However, whilst 5 needs five steps to reach 1, 6 needs eight ($6 \to 3 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1.$). Amazingly the chain starting with 27 requires one hundred and eleven steps to reach 1 and the chain has to go as high as 9232 (reached after seventy-seven steps) to do so!

The conjecture seems likely to be true; computers have checked starting values up to 2^{60} ($\approx 1.15 \times 10^{18}$) according to the Collatz Conjecture Wikipedia page. However, this is not a proof and there are two ways in which it could fail. There could be a chain out there in which the integers keep going higher and higher without limit and the chain just does not come down to 1. Alternatively, there could be a cycle of integers which therefore avoids going through 1.

In fact, I do know of four cycles:

- $0 \to 0 \to \ldots$
- $-1 \to -2 \to -1 \to \ldots$
- $-5 \to -14 \to -7 \to -20 \to -10 \to -5 \to \ldots$
- $-17 \to -50 \to -25 \to -74 \to -37 \to -110 \to -55 \to -164 \to -82 \to -41 \to -122 \to -61 \to -182 \to -91 \to -272 \to -136 \to -68 \to -34 \to -17 \to \ldots$

However, these four cycles do not concern us as they involve 0 or negative integers. Consequently, none of these contradict the Collatz Conjecture but they do make us wary enough to realise that cycles could be more than a mere theoretical possibility!

So, what can we say? Well, let us suppose, against the conjecture, that not all positive integers generate a chain that reach 1. The smallest such integer (which we will call S) must be of the form $4k+3$ (i.e. it has a remainder of 3 on division by 4). Why is this the case? Any positive integer must be of the form $4k$, $4k+1$, $4k+2$ or $4k+3$. If S were even, the first step would give us $\frac{S}{2}$ which is smaller than S. This would contradict the definition that S is the smallest integer generating a chain not reaching 1 as $\frac{S}{2}$ cannot reach 1 either being in the same chain. This means that S cannot be of the form $4k$ or $4k+2$.

Let us now consider $4k+1$:

$$S = 4k+1 \text{ (odd)} \to 12k+4 \text{ (even)} \to 6k+2 \text{ (even)} \to 3k+1.$$

If $k=0$ this would give the cycle $1 \to 4 \to 2 \to 1$.

If $k>0$, $3k+1 < 4k+1$ and so we have found an integer in the chain for S which is smaller than S which is again a contradiction.

Therefore. since S cannot be of the form $4k+1$, the only remaining form for S is $4k+3$ (if such a number exists at all which, of course, the Collatz Conjecture guesses it does not!)

Secondly, if there is a cycle it cannot involve multiples of 3. (You can look back at the examples of cycles I gave previously to notice that this was true in all these cases, apart from the trivial case of 0.) A positive integer which is multiple of 3 cannot be a power of 2, and so it is either odd already or if it is even there will come a point in its chain when an odd integer is reached. When you multiply by 3 and add 1, the next value is no longer divisible by 3. Moreover, halving a number does not suddenly make it divisible by 3, nor do further cases of multiplying by 3 and adding 1. Consequently, in any chain generated by a multiple of 3 all the values which follow after the first odd number cannot be divisible by 3. This means that we cannot return to the original number, which was divisible by 3.

This result can be combined with the previous one. We found that S, if it exists at all, must have a remainder of 3 on division by 4. This means that if we consider instead remainders after division by 12, S must have a remainder of 3, 7 or 11 i.e. $S = 12m+3$, $12m+7$ or $12m+11$ for some integer m. However, if S were the smallest integer in a cycle (note: S could avoid reaching 1 by starting a chain that has no upper limit in which case what follows does not apply), we have just learnt that S cannot be divisible by 3 as you would not be able to return to it. But $12m+3$ is divisible by 3. Therefore the smallest integer not reaching 1 must be of the form $12m+7$ or $12m+11$ if it is in a cycle.

Andrew Palfreyman

TREASURE HUNT – 17

An estimation treasure hunt question!

The closest submission to the correct number of plants in the red trapezium wins the prize.

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