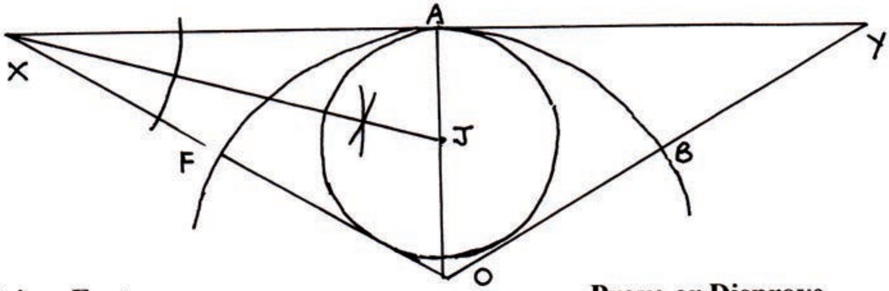
MAthematical PiE

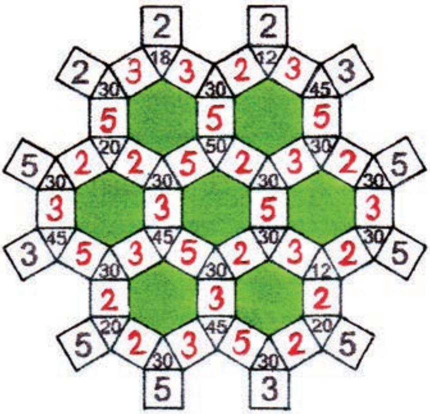
Spring 2021 **Notes for . . . . No. 212**

Poached Eggs Recooked

My plea for a simpler solution to the Poached Eggs problem in issue 210 was answered neatly by **Geoff Strickland,** who pointed out that each egg is the inscribed circle of a triangle such as XOY below. All that is necessary is to draw OFX = 2OF and XAY = 2XA, then the bisectors OA and XJ of angles FOB and AXO respectively:

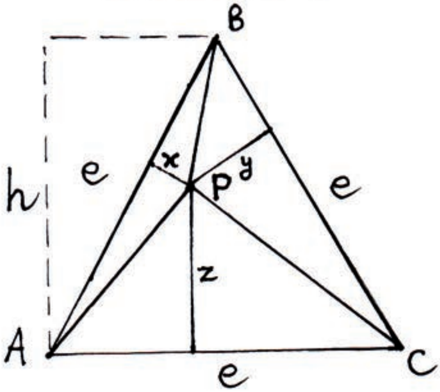


Prime Factors



This can be solved in one 'sweep' starting from any of the outside squares but two. Although starting from several is less work!

Prove or Disprove



With the labelling shown, the areas of the three triangles APB, BPC and CPA sum to the area of triangle ABC, $ex/2+ey/2+ez/2=eh/2$ so $x+y+z=h$. The editor tells me that this is known as *Viviani's Theorem.*

Fibonacci's Average Result

I like this for several reasons: **1.** Unusually, recourse to algebra does not initially give much help: ***a, b, (a + b)/2, (a + 3b)/4, (3a + 5b)/8, (5a + 11b)/16 . . . .???;*** **2.** 'Suck it and see' is vital - if you do not experiment, you get no flavour; **3.** Expression as decimals (using a calculator) makes the limit each time clear; **4.** Being systematic uncovers patterns; **5.** Using improper fractions enables us to unlock the pattern's secret.

Even trying a few starting pairs at random, two things are clear; every sequence (except the monotonous ***a,a,a,a,a*** .. types) oscillates up and down towards a limit, and many of the limits involve recurring 3's or 6's, suggesting that division by 3 is involved. You might even notice that the closer the two numbers chosen, the quicker the sequence converges.

Here are some results in a logical order:

1, 2, . .	1.666... or 5/3	2, 1, . .	1.333... or 4/3	3, 1, . .	1.666... or 5/3
1, 3, . .	2.333... or 7/3	2, 3, . .	2.666.. or 8/3	3, 2, . .	2.333... or 7/3
1, 4, . .	3.0 or 9/3	2,4, . .	3.333... or 10/3	3. 4, . .	3.666... or 11/3

It would seem that ***a, b, . .*** approaches ***(a + 2b)/3.*** Test this by predicting, say, that 4, 5, . . should approach 14/3 or 4.666... Calculation confirms this. Expressing the successive terms as improper fractions helps us to explain it: 4, 5, 9/2, 19/4, 37/8, 75/16, 149/32, . . .

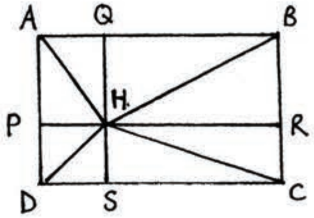
Find the difference between 14/3 and terms after the first two:

$14/3 - 9/2 = (28 - 27)/6 = 1/6$ $\quad$ $14/3 - 19/4 = (56 - 57)/12 = -1/12$

$14/3 - 37/8 = (112 - 111)/24 = 1/24$ $\quad$ $14/3 - 75/16 = (224 - 225)/48 = -1/48$

This shows both the oscillation and the consistent reduction of the difference. We can get as close as we like to the predicted limit by going far enough along the sequence. This is the very definition of what a limit is.

The procedure used above on 4, 5, . . can be generalised for the algebraic sequence ***a, b, . .*** to prove that ***(a +2b)/3*** is the correct limit.



Logan and Heidi

Draw lines PR and QS through H parallel to the sides of the rectangle.

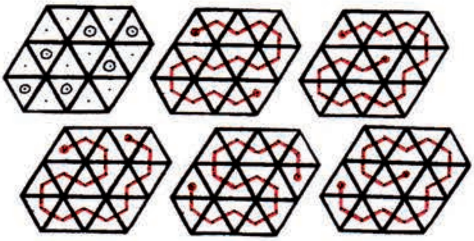
$AH^2 + CH^2 = AP^2 + PH^2 + HS^2 + SC^2$

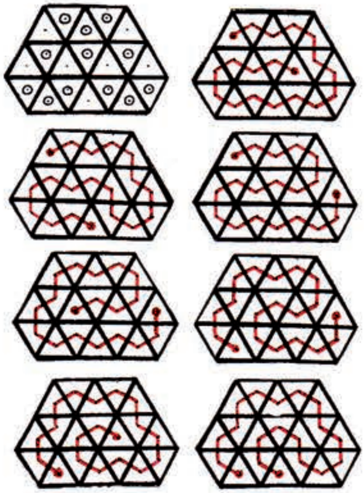
$= BR^2 + DS^2 + HS^2 + HR^2$

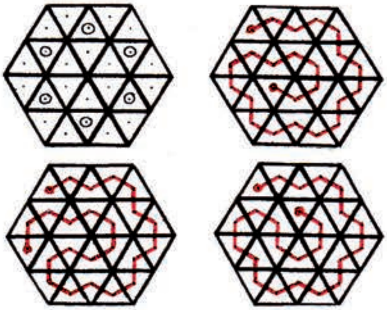
$= (BR^2 + HR^2) + (DS^2 + SH^2) = BH^2 + DH^2$ This will be true wherever H is.

Count on Us!

The diagrams opposite show by circles which triangles cannot be starting points. It would seem perverse not to rule out symmetries! Most of the paths can be numbered from either end to give two numberings.





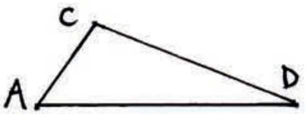


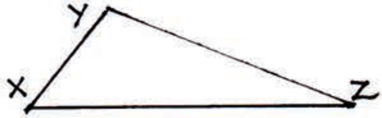
There are a) 5 paths giving 8 (sic) numberings, b) 3 paths giving 6 numberings, and c)7 paths giving 14 numberings.

None of the paths end with the last digit next to the first. The impossibility of such 'closed' paths can sometimes be proved by imagining colouring the grid like a checkerboard. Can you see why? Which of the grids can this proof be applied to?

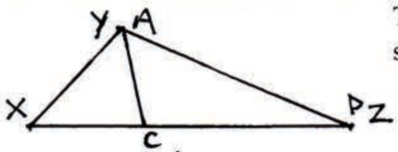
A Straight Line?

It is essential not to assume what we are trying to prove: do not regard ABCD as a triangle! One method would be to apply the cosine formula to each of the triangles ACB and ACD, showing that cos ACB = 5/16 and cos ACD = – 5/16. I prefer:





Triangles ACD and XYZ drawn above are similar, in the ratio 4:5, so they are equiangular. Since AD =YZ the smaller triangle will fit into the larger like this:

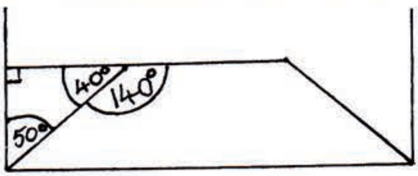


Then XC = 25 – 16 = 9

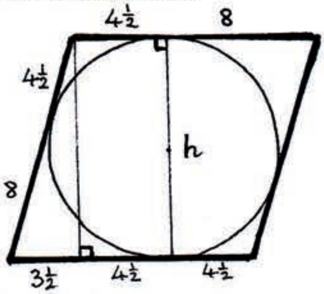
so triangle AXC is congruent to ABC.

When? In 2070: $45^2 + 45 = 2025 + 45$ **The Dice:** the band of three-spots.

The Nonagon and the Square



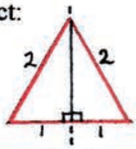
Not Much Data?

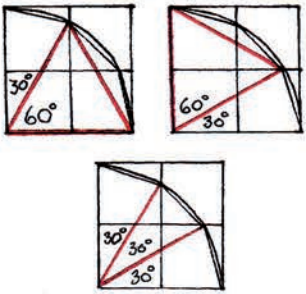


The height ***h*** is given by
$h^2 = 12.5^2 - 3.5^2 = 16 \times 9$
so $h = 4 \times 3 = 12$ and the area
of the rhombus is $12.5 \times 12 = 150$

Jack and Jill

Jill is correct:





Sharing the Jackpot

$997920 = 2^5 \times 3^4 \times 5^1 \times 7^1 \times 11^1$ so it can be shared $6 \times 5 \times 2 \times 2 \times 2 = 240$ ways. Finding appropriate factors of 240 that lead to amounts below 1m is not easy!

$5 \times 3 \times 2 \times 2 \times 2 \times 2$ leads to $2^4 \times 3^2 \times 5 \times 7 \times 11 \times 13 = 720720$

$5 \times 4 \times 3 \times 2 \times 2$ leads to $2^4 \times 3^3 \times 5^2 \times 7 \times 11 = 831600$

$8 \times 5 \times 3$ leads to $2^7 \times 3^4 \times 5^2 = 259200$ $8 \times 3 \times 5$ leads to $2^7 \times 3^2 \times 5^4 = 720000$

The Green Block of Wood $3 \times 7 \times 9 - (3-2)(7-2)(9-2) = 189 - 35 = 154$

Backwards in time 11.45 **From *Puzzle Papers*** 15 (triangular numbers)

Flipping Discs Put 10 behind 11 and 11 behind 9 or 13 behind 11 and 8 behind 9.

Didgetty Fidgetty

■	3	4	5	6	■	2	4	6
■	8	■	■	1	2	3	■	2
6	4	■	2	2	■	4	1	4
3	■	5	4	■	4	5	■	■
6	3	0	■	■	■	6	0	6
■	■	3	6	■	1	7	■	8
5	6	7	■	4	8	■	1	2
1	■	4	0	9	■	■	0	■
2	8	8	■	5	0	4	0	■

Can You Solve It?

Using properties of chords (or the similar triangles they are based on), the remaining part of the diameter is given by $12^2/6 = 24$.

The area is $(15^2 \times \pi)/2 = 112.5\pi$

A Quick Student 2: