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A Bridge Too Far:
The Perambulators of Königsberg (and Other Cities)

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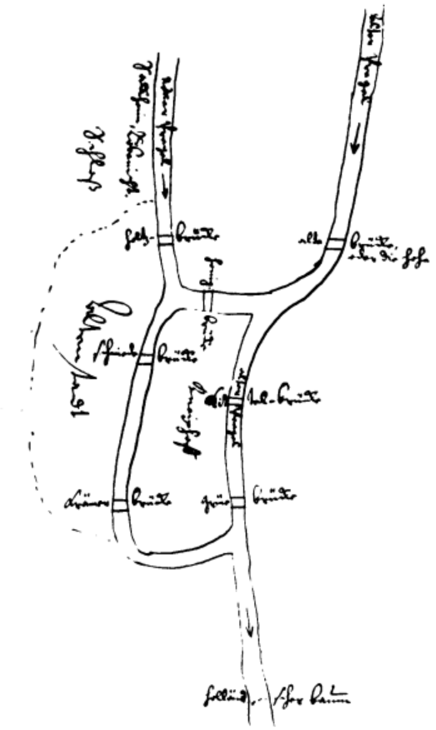
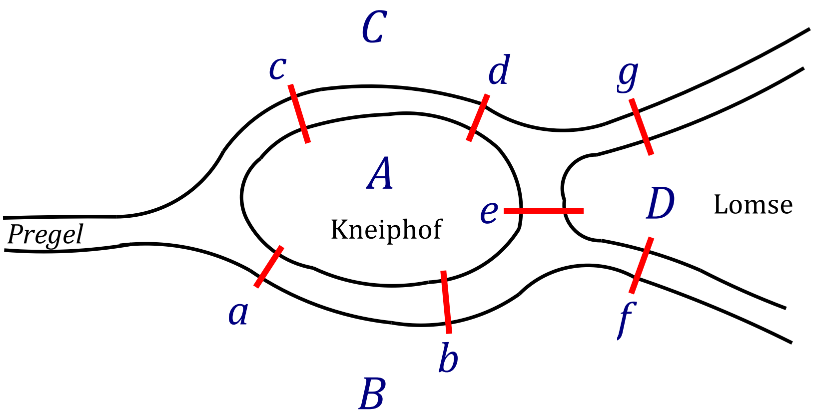
In March 1735, the mayor of Danzig (now Gdańsk), wrote to Leonhard Euler (1707–1783) with a question that had emerged from the city of Königsberg, a hundred miles away. The mayor, Carl Gottlieb Ehler (1685–1753), had already had discussions with a mathematician born in Königsberg, Heinrich Kühn (1690–1769), then living in Danzig and soon to become professor at the city's Academic Gymnasium. They wanted to know why it was that the Königsbergers could not find a route around their city and back to their homes that crossed each of its seven bridges just once. A sketch of the river and its bridges was included for Euler's convenience (shown alongside; see Sachs et al, 1988).



Leonhard Euler had arrived in St Petersburg in 1727, initially to become a professor of physics but by the time the letter from Ehler arrived he held the senior chair in mathematics. His initial thoughts were that the "question was so banal" and yet still "worthy of attention in that neither geometry, nor algebra, nor even the art of counting was sufficient to solve it". He recognised it as belonging to the geometry of position, which had been alluded to by Leibniz and he said so when he rose to speak at the St Petersburg Academy of Sciences on 26 August 1735 and indeed in the paper published in its proceedings the following year. (A translation is available in Biggs, Lloyd & Wilson, 1976.)

Königsberg, then a Prussian port, and now Kaliningrad in Russia, sits on the River Pregel in which lie the islands of Kneiphof and Lomse. These islands taken with the north and south banks of the river constitute four areas of land which Euler labelled A, B, C and D. The bridges he marked with the lower case letters from a to g.

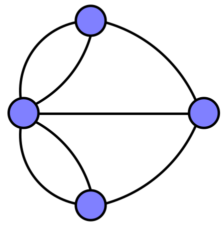


Whether a return to the starting point or not was envisaged, if it were possible to cross each bridge once, there should be a string of upper case letters that would encapsulate the journey. If all seven bridges were to be crossed then that string would consist of eight letters and if the circuit were complete, then the first and last letters would coincide.

Euler argued that since A is connected by five bridges, it must be visited three times (onto **and** off the island twice and onto **or** off the island once). The other three areas of land are reached or left via three bridges and hence must be visited twice. So in total nine upper case letters are needed to represent a journey around the bridges, though a route over them requires just eight. In effect, Euler had outlined a proof by contradiction.

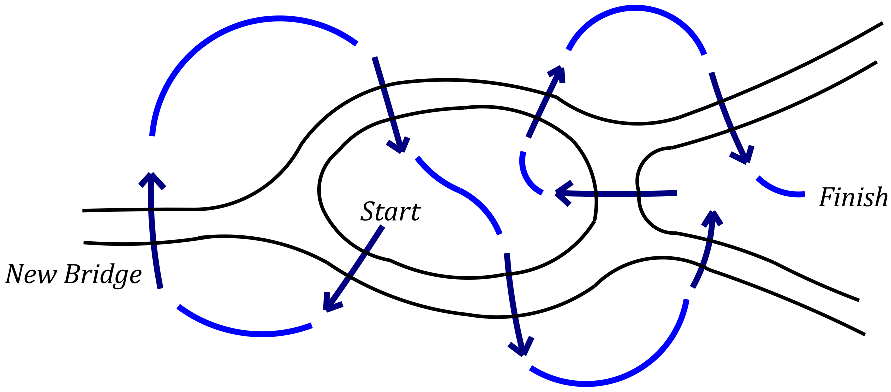
Euler noted that if the number of bridges connecting an area is odd, then the number of times that area is visited is half of one more than the number of bridges. If the number of bridges connecting an area is even, then the number of times that area is visited is simply half the number of bridges. There is an efficiency about the 'evens' than is missing from the 'odds'.

To use modern terminology, for an 'Eulerian circuit', each bridge is crossed exactly once, with the starting and finishing points being one and the same. Euler proved that for this to be possible, all 'vertices' need to be of even degree; that is, there is an even number of connecting 'edges'. The graph of such a circuit is called an 'Eulerian graph'. For an 'Eulerian path', each bridge is crossed exactly once but the starting and finishing points differ. If no vertices are odd, then the path is also a circuit; otherwise the path needs to have two odd vertices and its graph is termed semi-Eulerian. Eulerian paths and hence Eulerian circuits are 'traversable'. The graph representing the Königsberg bridges problem would look like this:



The Königsberg bridges reimagined at the University of Canterbury, Christchurch, New Zealand

In 1875, a new bridge was built across the Pregel, providing the first direct link between the north and south banks (Euler's vertices B and C). This changed the parity of those two vertices from odd to even. The eight-bridge city can be represented by a graph with two even and two odd vertices. From that time onwards, the perambulators of Königsberg could cross each of the bridges in succession, though, to their continuing frustration, that still left them one bridge from home. Here's such a route.



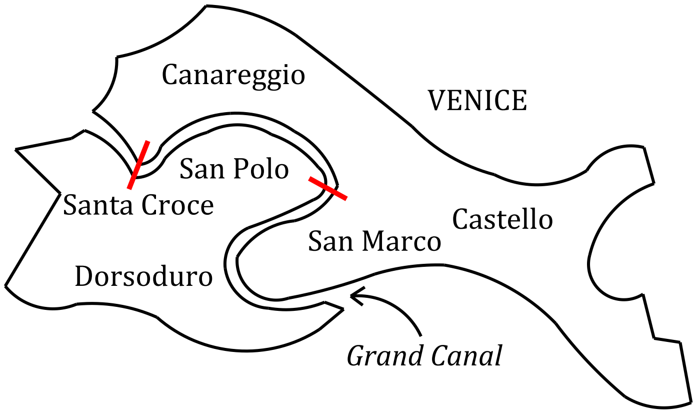
References

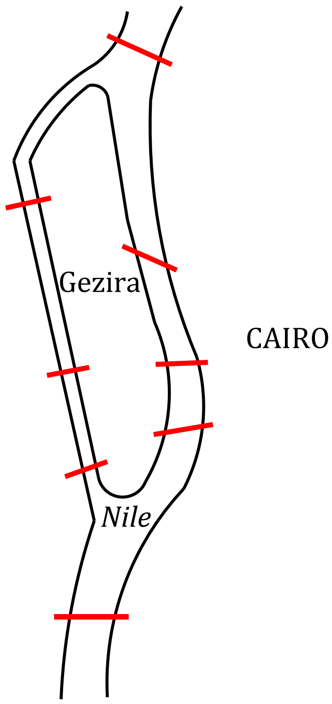
Biggs, N. L., Lloyd, E. K. and Wilson, R. J., *Graph Theory 1736–1936*, Clarendon Press, Oxford, 1976, reprinted 1998.

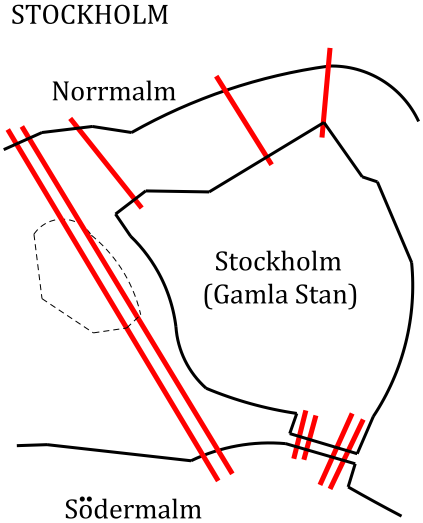
Sachs, H., Stiebitz, M. and Wilson, R. J., 'An historical note: Euler's Königsberg letters', *J. Graph Theory* 12 (1988), 133-139.

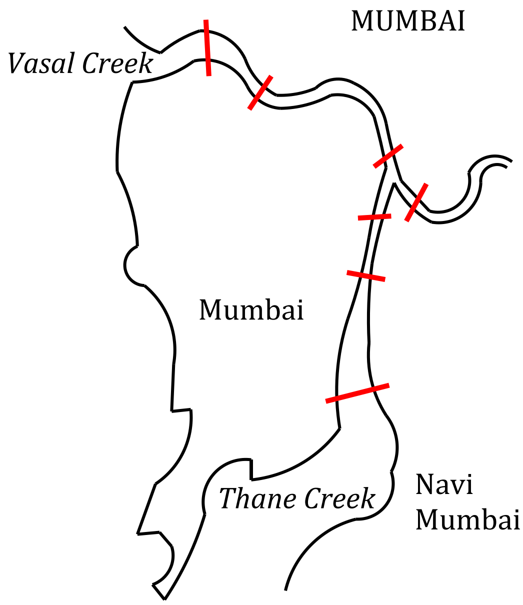
Perambulating modern-day cities

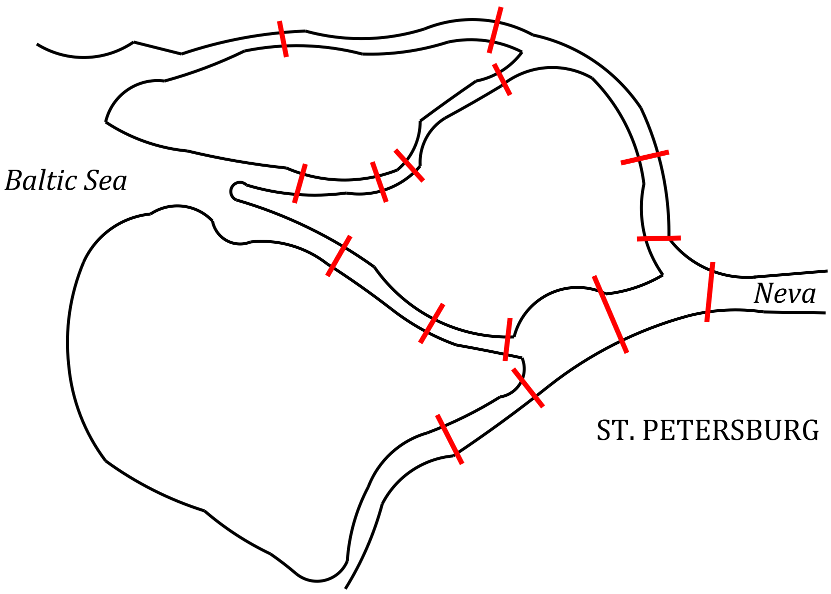
How would the perambulators of these eleven cities get on?

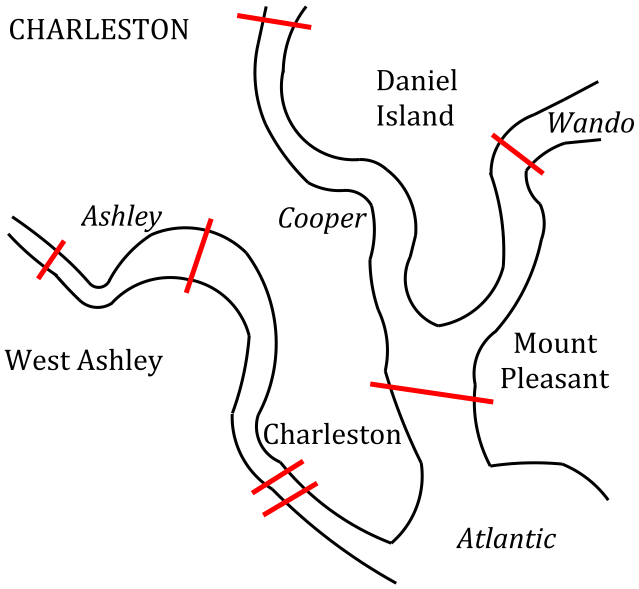


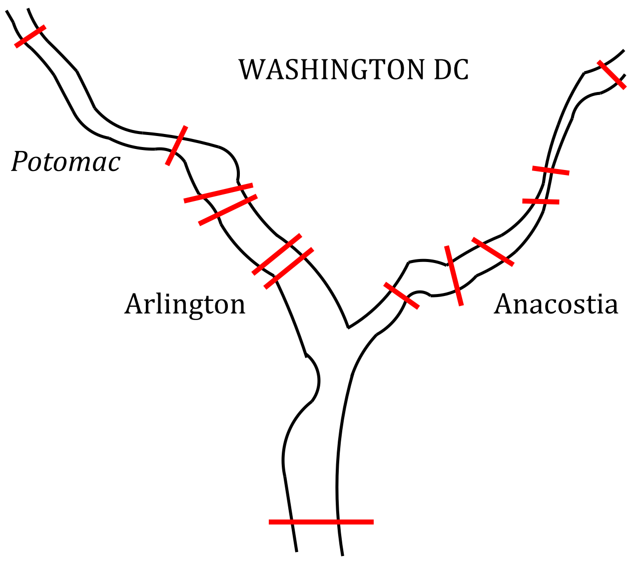


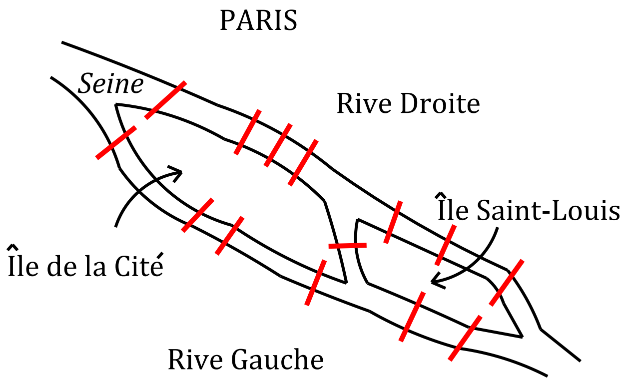


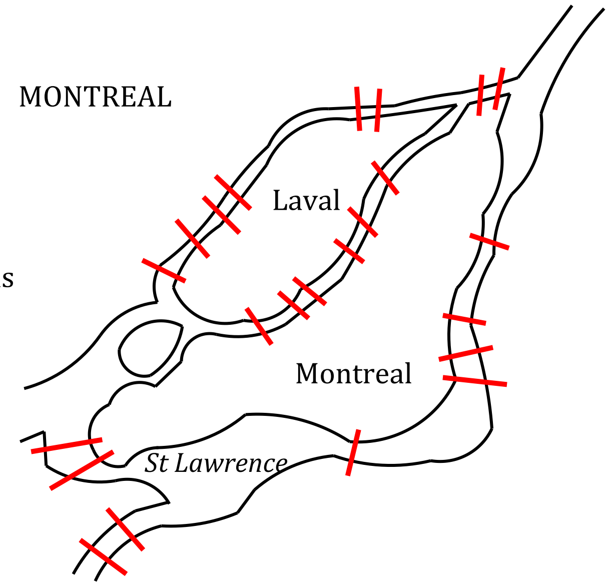


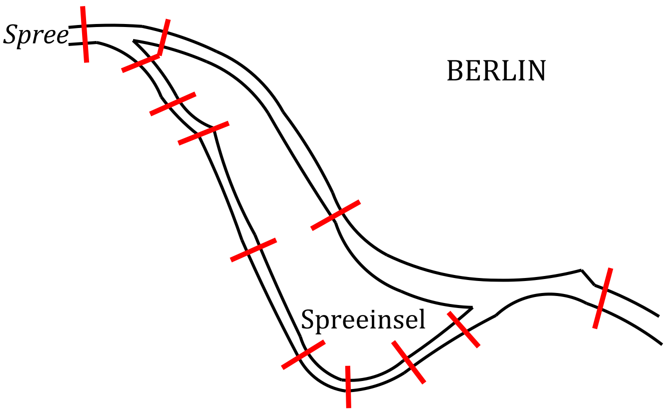

Baltic Sea
Neva
ST. PETERSBURG

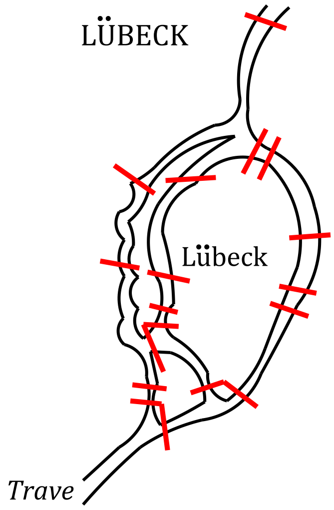

CHARLESTON
Daniel Island
Wando
Ashley
Cooper
West Ashley
Mount Pleasant
Charleston
Atlantic


WASHINGTON DC
Potomac
Arlington
Anacostia







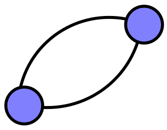


Answers

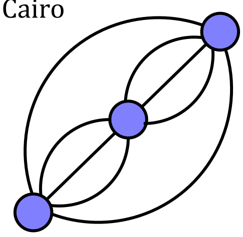
All cities are traversable but a circuit is not available in every case.

City	No. of even vertices	No. of odd vertices	Classification
Venice	2	0	Eulerian
Cairo	1	2	Semi-Eulerian
Stockholm	1	2	Semi-Eulerian
Mumbai	3	0	Eulerian
St Petersburg	3	2	Semi-Eulerian
Charleston	4	0	Eulerian
Washington	1	2	Semi-Eulerian
Paris	2	2	Semi-Eulerian
Montreal	3	2	Semi-Eulerian
Berlin	3	0	Eulerian
Lübeck	1	4	Semi-Eulerian

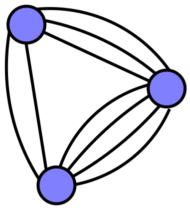
Venice



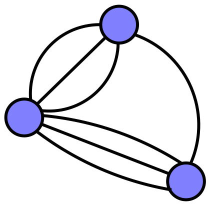
Cairo



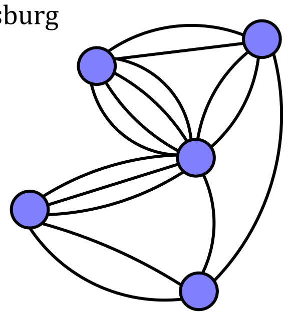
Stockholm



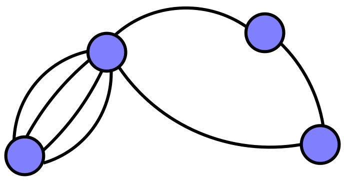
Mumbai



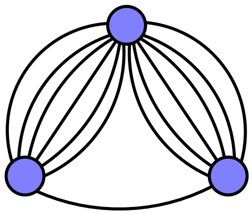
St Petersburg



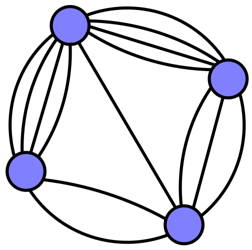
Charleston



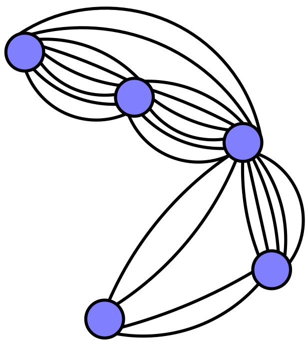
Washington



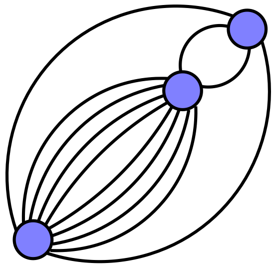
Paris



Montreal



Berlin



Lübeck