PRESIDENTIAL ESSAY NO. 5: AUGUST 2021

Source Code: Exploring Binary Bubbles

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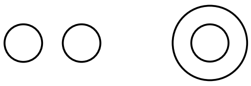
Supporting mathematics in education

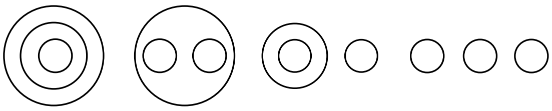
There was a time when binary was commonly taught in mathematics and computing classes. Indeed, bases other than binary were also taught under the 'modern mathematics' that was in curricular vogue from the 1960s onwards. As for contexts, we were largely restricted to binary strings in computing but there are interesting and accessible contexts for binary that are still poorly known, including this very visual pattern-generating example.

The notion of 'bubble patterns' was floated by David Wells and used to fine pedagogical effect by Don Steward (see the May issue of *Mathematics in School*). In its explanatory and predictive powers, the approach that follows goes beyond the outcomes from Don's investigation.

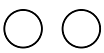
Given two closed loops ('bubbles' drawn as circles here) which neither touch nor intersect, there are two different ways of drawing them — alongside each other in the first configuration and one inside the other in the second:

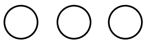


It does not matter how large one bubble is relative to another, nor are their relative positions important, though the inside/outside concept is fundamental. If we extend the idea to three bubbles, there are four different configurations:

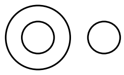


For four bubbles there are nine patterns and you'll probably have no problem finding them in 5 minutes. Of course, as the number of bubbles increases, it becomes trickier to find all the patterns. You'll need to think about the strategies you are using, perhaps including adding extra bubbles in different ways to some of patterns already discovered. For example, if we add an extra bubble to

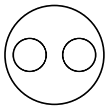


we can get  .

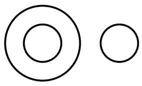
But by putting it outside (or inside) a bubble, we have



and by wrapping it around both, we have



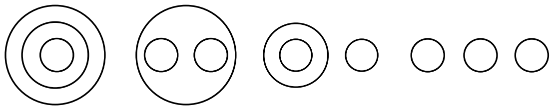
Here's a way of generating the patterns using binary notation. Take the third of the patterns for three bubbles:



and crop it at the top and bottom:

(( )) ( )

In appearance, this gives something like a sequence of brackets: (( )) ( ). The other three configurations of three bubbles give, respectively: ((( ))), (( ) ( )), ( ) ( ) ( ). And we can take this coding further, using a 0 for an opening bracket and a 1 for a closing bracket. Then the three-bubble patterns, bracket sequences and binary codes are:



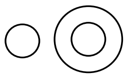
((( )))	(( ) ( ))	(( )) ( )	( ) ( ) ( )
000111	001011	001101	010101

Since every sequence of brackets begins with an opening bracket and finishes with a closing bracket, perhaps they could be ignored, as could the first and last digits in the binary codes, which then become:

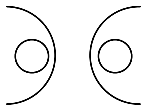
0011 0101 0110 1010

These four abbreviated codes each have two zeros and two 1s. (Every bracket, once opened, has to be closed.) But there are two other binary codes consisting of two zeros and two 1s: 1001 and 1100. Can bubble patterns relating to these abbreviated binary codes be generated by working in reverse, and if so, what would they look like?

1 0 0 1
( ) (( ))

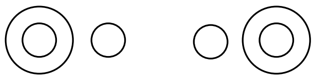


1 1 0 0
( )) (( )



Each of these designs begs a question:

1. For the first part of our discussion we would surely have considered the two bubble configurations below as one and the same, but is there a connection between their binary codes?



2. Can we incorporate the impossible or 'burst' bubble diagram in our analysis?

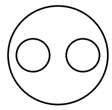
To answer the first, we must use the full, rather than the abbreviated, binary codes and make a comparison (see next column). The association is revealed by reversing the order of the binary digits and then taking complements (swapping zeros and 1s).

(( )) ( ) 001101

( ) (( )) 010011

If we carry out the same operations on the other three binary sequences, they do not change. That there is a reflective property is apparent and it is also obvious from the bubbles.

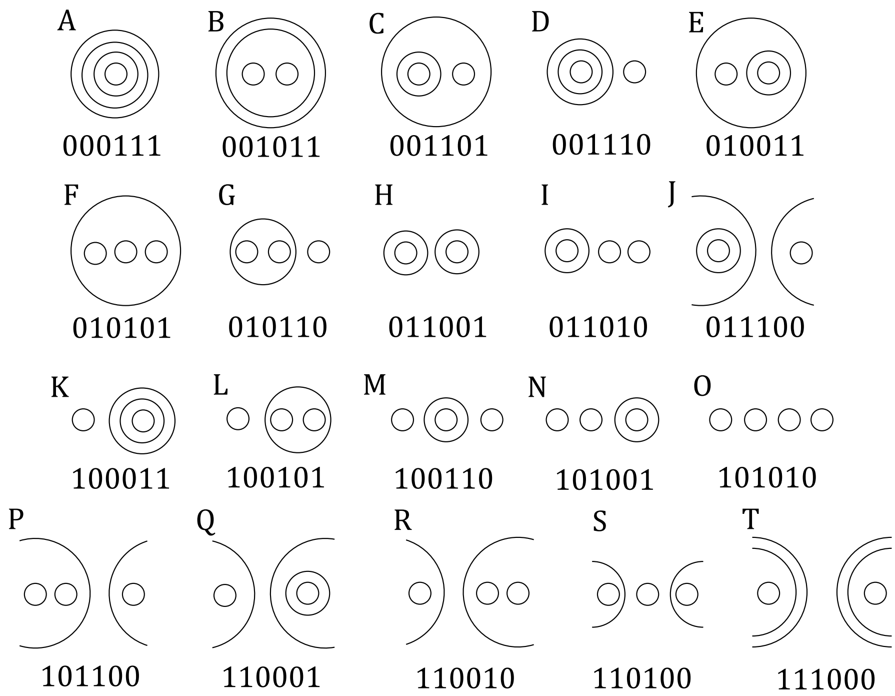
If we are drawing our diagrams on an ordinary sheet of paper, the burst bubble should be discarded. But we can incorporate it if we think of it drawn on the curved surface of a cylinder, for example. (Think of the label on a tin or beans!) Then, from an appropriate vantage point, we would see:



We can see the relationship between the burst bubble and the button design above. It is cyclical in nature:

Button 011001 ↔ 001011 Burst

For four bubbles, it's worth considering how many patterns we are seeking. The abbreviated code consists of 6 digits, three zeros and three 1s. There are $\binom{6}{3} = 20$ such patterns, though not all define valid bubbles.



The figure above contains all 20 bubble diagrams, in order of their binary code, and labelled A to T for convenience. Numerous observations might be made:

- Six bubbles are burst. They are J, P, Q, R, S, T. S and T are symmetrical, while the remaining four constitute two sets of twins: (J, Q), (P, R).
- Of the 14 valid bubbles, six are symmetrical (A, B, F, H, M, O) and there are three sets of twins, (C, E), (D, K) and (G, L).
- (I, N) is also a reflective pair but both are variants of the symmetrical M and should be discounted. Or to put it another way, M, I and N form a triplet.

So in total, there are nine different, valid bubble designs, represented by (for example) A, B, C, D, F, G, H, M and O.

Task: Five-bubble designs

(Designed to be given as a homework task for individual pupils or as a group task in class.)

Find all the different five-bubble designs, identifying symmetrical bubbles, bubbles that have reflective twins and burst bubbles and make observations, frame conjectures and draw conclusions throughout.

Possible Solution

The number of possible bubbles is the number of sequences of eight digits, four 1s and four zeros,

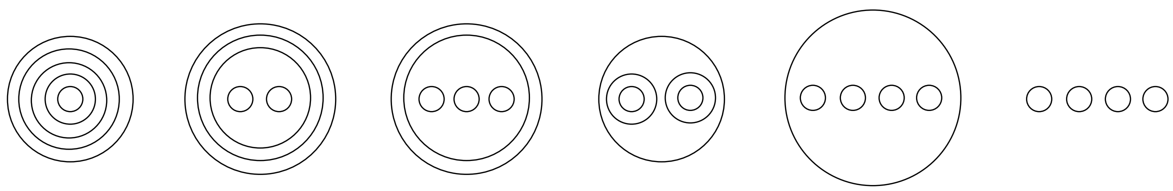
$$\binom{8}{4} = 70.$$

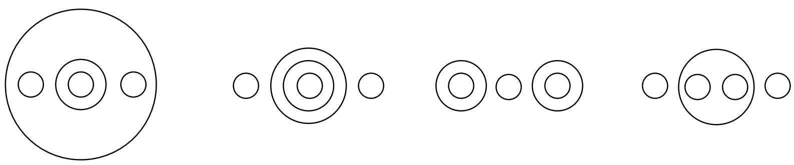
Here is a table of all 70, in order.

00001111	00111100	01101001	10011001	11000101
00010111	01000111	01101010	10011010	11000110
00011011	01001011	01101100	10011100	11001001
00011101	01001101	01110001	10100011	11001010
00011110	01001110	01110010	10100101	11001100
00100111	01010011	01110100	10100110	11010001
00101011	01010101	01111000	10101001	11010010
00101101	01010110	10000111	10101010	11010100
00101110	01011001	10001011	10101100	11011000
00110011	01011010	10001101	10110001	11100001
00110101	01011100	10001110	10110010	11100010
00110110	01100011	10010011	10110100	11100100
00111001	01100101	10010101	10111000	11101000
00111010	01100110	10010110	11000011	11110000

The 28 sequences highlighted in turquoise represent burst bubbles. They are those starting with two 1s or with three 1s in the first four or with four 1s in the first six (or those ending with two zeros). This leaves 42 binary sequences for which bubble arrangements are possible.

These valid sequences fall into four subsets. The first two subsets have 10 distinct members altogether (highlighted in yellow above); they represent symmetrical bubble diagrams — when looked at in a mirror or from behind they remain unchanged. But these symmetrical members are of two types. Six of them are of a simple, singlet form (upper row, below) and the remaining four are found in triplets (lower row).

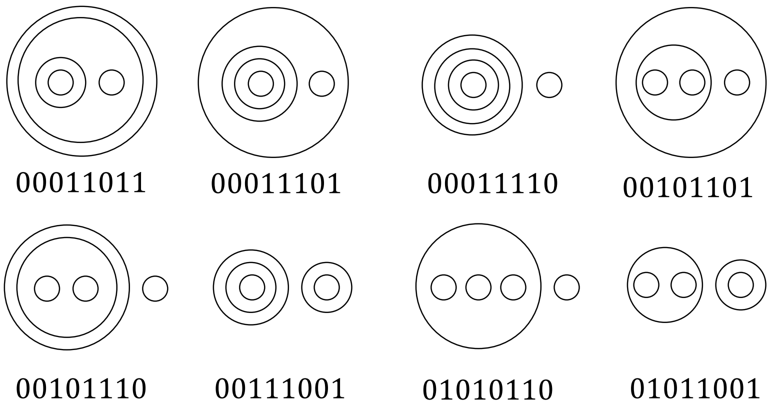




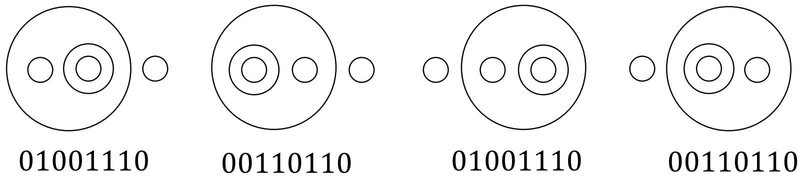
The remaining 32 binary sequences have a pairwise reflective property. There are 8 sets of twins (highlighted in green) and two sets of quadruplets (highlighted in purple and turquoise).

00011011, 00100111 00011101, 01000111 00011110, 10000111 00101101, 01001011	00101110, 10001011 00110101, 01010011 00110110, 10010011 00111001, 01100011	00111010, 10100011 01001110, 10001101 01010110, 10010101 01011001, 01100101	01011010, 10100101 01100110, 10011001 01101010, 10101001 10011010, 10100110

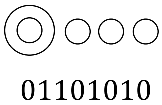
The first of each pair of these 8 sets of twins is drawn below.



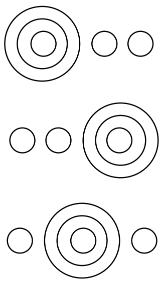
The first set of quadruplets is shown below.



The other set is represented by:



Each of the pairs with no highlighting are versions of one of the symmetrical designs already drawn. For example, 00111010 and its reflection, 10100011, belong to 10001110.



So there are four triplets (not four additional pairs).

So altogether, we have 6 symmetrical singlets, 4 symmetrical triplets, 8 sets of twins and 2 sets of quadruplets. That's 20 distinct, valid bubbles.

Note

This is a revised version of 10 August 2021, the revisions necessitated by the helpful observations of the Hungarian mathematician, Zoltán Retkes (now resident in Barwell, Leicestershire). Zoltán has also drawn my attention to the connection between my results and both Catalan and Narayana numbers. He comments:

> When you count the valid configurations you may classify the bracket sequences according to the number of nestings, e.g., if $n = 4$ then (()(())) contains two nestings while (((()))) contains one. The number of valid sequences with k nestings is given by the Narayana number $N(n, k)$ for n pairs of brackets. In this example $N(4,1) = 1$, $N(4,2) = 6$, $N(4,3) = 6$, $N(4,4) = 1$. The sum of these numbers is $1 + 6 + 6 + 1 = 14$, the fourth Catalan number. For $n = 5$ you would have $1 + 10 + 20 + 10 + 1 = 42$, the fifth Catalan number and so on.

Reference

Pritchard, C., 'Binary bubbles', *SYMmetry*plus 50 (Summer 2013), 3-4, 15.