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A Shot in the Dark: Exercises in Visualisation

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Sir Michael Atiyah, Fields Medallist in 1966, was one of the great mathematical thinkers of the last century. In 1981-82 he was President of The Mathematical Association and so it was that in the spring of 1982 he gave his Presidential Address on 'What is geometry?' The presentation was published in the *Mathematical Gazette* (Atiyah, 1982). Here is part of what he said:

> …the commonest way to indicate that you have understood an explanation is to say "I see". This indicates the enormous power of vision in mental processes, the way in which the brain can analyse and sift what the eye sees. Of course the eye can sometimes deceive and there are optical illusions for the unwary but the ability of the brain to decode two- and three-dimensional patterns is quite remarkable.
>
> Sight is not however identical with thought. We have trains of thought which take place in sequential form, as when we check an argument step by step. Such logical or sequential thought is associated more with time than with space and can be carried out literally in the dark. …
>
> Broadly speaking I want to suggest that geometry is that part of mathematics in which visual thought is dominant whereas algebra is that part in which sequential thought is dominant. This dichotomy is perhaps better conveyed by the words "insight" versus "rigour" and both play an essential role in real mathematical problems.
>
> The educational implications of this are clear. We should aim to cultivate and develop both modes of thought.

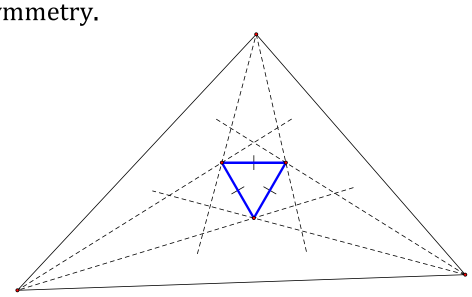
A couple of decades later, Douglas Hofstadter sent an email to Michael Atiyah about his address, relating what Atiyah had written to his own experience. Doug and I had been in contact since he had written the foreword to my book *The Changing Shape of Geometry* in 2002 and so he copied me in. (For anyone unfamiliar with Hofstadter, at the time he held chairs in both Cognitive Science and the History and Philosophy of Science at Indiana University but was on sabbatical at the University of Bologna; he had come to prominence earlier by succeeding Martin Gardner as the mathematics columnist for *Scientific American*, for writing a number of books, notably *Gödel, Escher, Bach*, and for being a polyglot polymath with an ability to see connections between disparate material.) With the permission of both parties the email was reproduced in *Mathematics in School* (Hofstadter, 2007).

Hofstadter began by alluding to Atiyah's comment about thinking in the dark and recalling that the Berlin-based Swiss geometer, Jakob Steiner (1796–1863), "was so opposed to the use of diagrams in geometry that he insisted on teaching all of his geometry classes literally in the dark … He would cover all the windows of his room and would force his students to think in the dark". So Hofstadter tried it himself:

> I was teaching a geometry course that met at my house in the evenings. One day I realized that the room in which we met could easily be rendered pitch-dark. Then one evening, quite on a lark, I did just that, and then I proceeded, totally spontaneously, to describe Morley's theorem relating a full proof of it to the class members, entirely in words, in pitch-dark. Of course as I spoke to them I made gestures galore with my arms, but naturally my students saw nary a one of them – my gestures were solely for my own comfort in self-expression. I believe I did a pretty good job of it, and that the students digested the proof very well – perhaps just as well as if I had done it in a series of diagrams before their eyes.
>
> Anyway, teaching geometry in a pitch-dark room is a wonderful exercise both for teacher and for students, and it of course forces one to ponder, "What is visual imagery, if it is not seen by the eye?" It makes one realize that blind people have visual imagery every bit as rich as that of sighted people, and that indeed the eyes are just the entry channels for visual imagery for sighted people but that the actual imagery transcends vision and has to do with how space and shapes in space are represented in the brain, and that that has nothing intrinsic to do

with the entry channels. Such internal representations are necessarily built up by beings that have to negotiate 3-space as they live. Thus bats have visual imagery, blind people do, and so forth. Eyes are not needed for visual imagery, though they help.

It seems to me that teaching Morley's Theorem would be a step too far in schools (in perfect light or otherwise). The theorem says that if we begin with an arbitrary triangle and trisect the angles, the lines produced intersect at the vertices of an equilateral triangle, producing symmetry out of asymmetry.



Morley's Theorem

But if this is pushing things, what might reasonably be done? Here are a few ideas. These questions could be given in written form or simply read to a class (perhaps twice or even three times). Pupils are allowed to write down anything short of drawing a figure but are encouraged to write nothing at all. They are required to evoke a relevant image, work out the answer and explain how they reached it.

Warm-up

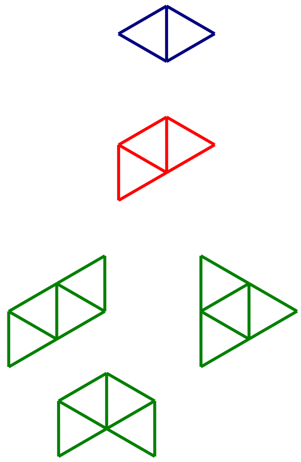
A. What shapes can be made from 2 equilateral triangles, 3 equilateral triangles, 4 equilateral triangles, stuck together edge to edge?

B. 30-60-90° triangles: what shapes can be made from two of them?

C. Imagine taking a piece of card in the shape of a regular pentagon sitting on one side. Cut it vertically along its line of symmetry. Take the right-hand piece of card and flip it over so that what was at the bottom is now at the top, and reattach it to the left-hand piece. What shape do you have now? If the original pentagon had area A what is the area of the new shape? If the pentagon had perimeter P, what is the perimeter of the new shape?

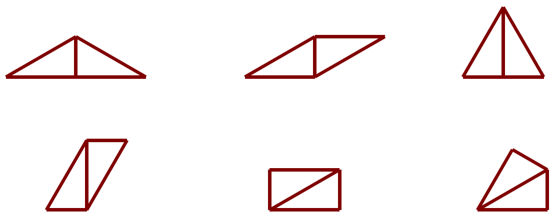
Answers

A.



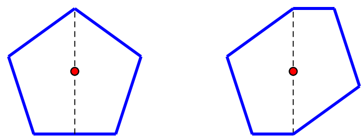
Rhombus; isosceles trapezium; parallel-gram, equilateral triangle and hexagon with bilateral symmetry.

B.



Two isosceles triangles (one of them equilateral), two paralleograms, rectangle, kite.

C.



Neither the area nor the perimeter of the pentagon change as we move to a hexagon with rotational symmetry of order two.

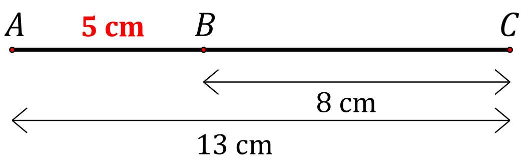
Main questions

1. Three points, A, B and C, lie on a straight line. If $BC = 8$ cm and $AC = 13$ cm, what is the length of AB?
2. Four points (A to D) are arranged symmetrically on a straight line. If $AC = BD = 15$ cm and $AD = 21$ cm, how far apart are B and C?
3. The front of a tower block is rectangular, its height greater than its width. Imagine a line drawn from the bottom left corner (vertex) to the top right corner. What can you say about the angle between that line and the base of the rectangle?

4. XYZ is a triangle with base XY. Angle $ZXY = 50°$ and angle $XYZ = 70°$. A line is drawn vertically from Z until it meets the base XY at P. Is P nearer to X or Y ?
5. EFG is a triangle with base $EF = 10$ cm. The lengths of EG and FG are 6 cm and 8 cm respectively. A perpendicular from G meets the base EF at H. Is H nearer to E or F ?
6. An isosceles triangle, with a vertical line of symmetry, has base 9 mm. A straight line is drawn from one end of the base to the midpoint of the opposite side; a second line is drawn from the other end of the base to the midpoint of its opposite side. How far apart are the midpoints?
7. An isosceles trapezium $ABCD$ has base $AB = 2$ cm, with the remaining sides all of length 1 cm. What is the size of angle DAC ?
8. A circle is trapped inside a domino shape, 20 cm long and 10 cm high. It touches the left-hand side, the top and the bottom. How far is the centre of the circle from the right-hand side of the domino?
9. Now, still imaging the figure for the previous question, a smaller circle is drawn, its radius only half that of the first circle. It touches the bottom of the domino and the lowest point of the first circle. How far is the second circle from the right-hand side of the domino at its nearest point?
10. ABC is a right-angled triangle with horizontal base AB and vertical side BC. $\angle CAB = 30°$ and $\angle BCA = 60°$. The angles at A and B are bisected and the lines created meet at I. What is the size of angle AIB ?
11. Three identical 2×2 squares are arranged with two touching side to side on the bottom and the third symmetrically placed on top of them. A line is drawn from the bottom-left corner of the bottom-left square to the top-right corner of the upper square. How long is it?

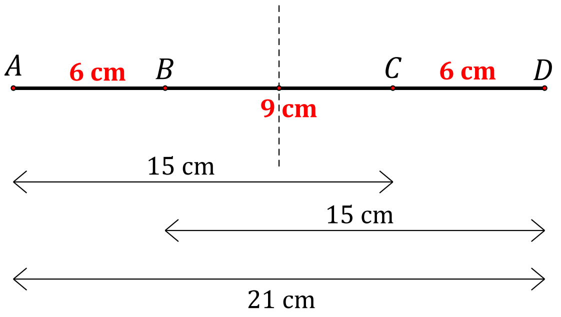
Here are possible diagrams and solutions.

1.



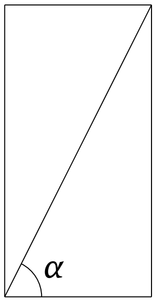
A simple subtraction gives the answer of 5 cm.

2.

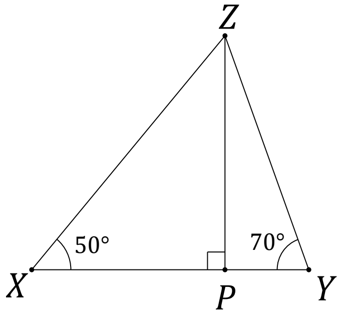


By subtraction (twice),
$AB = CD = 21 - 15 = 6$ cm.
So $BC = 9$ cm.

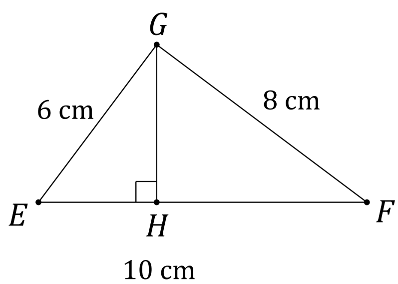
3. The only thing we can say is that the angle (α below) is greater than 45°.



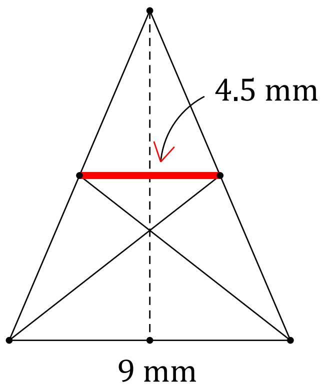
4. Because YZ is steeper than XZ, P is nearer to Y than to X.



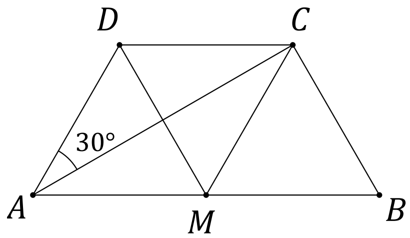
5. Because EG is shorter than FG, H is closer to E than F.



6. The midpoint theorem says that such a line has half the length of the base.



7. The isosceles trapezium can be split into three equilateral triangles. AC bisects $\angle DAM$, so $\angle DAC = 30°$.



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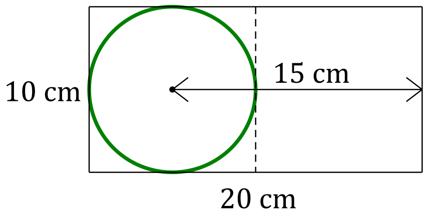

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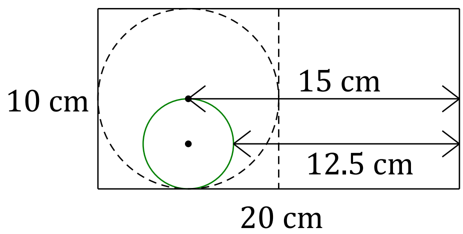

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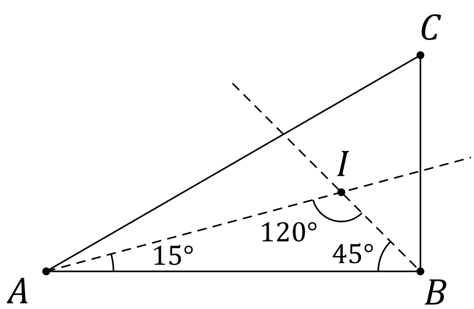
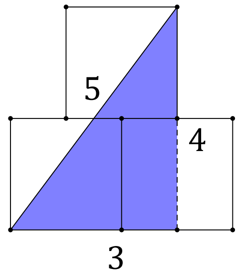


11. Produce a right-angled triangle of base 3 and height 4. The required length of 5 follows by Pythagoras' Theorem.



References

Michael Atiyah, 'What is geometry', *Mathematical Gazette* Vol. 66, No. 437 (October 1982), 179-184.

Douglas Hofstadter, 'Thoughts on geometrical thinking', *Mathematics in School* 36, 4 (September 2007), 27.