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The Domino Principle: Solomon Golomb's Polyominoes

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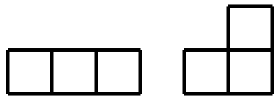
Introduction

How do you keep your teaching fresh? How do you ensure that your pupils are alert, keen to be in your class and making connections between parts of the subject that may appear distinct and disjoint? Of course there are some elements of the curriculum that are fairly routine, some perhaps even a little dull, but there are others that benefit from an appeal to the visual or that have a strong exploratory nature. And then there are those extracurricular inputs that light up the subject and the pupils themselves. Early in my career I came across Martin Gardner's books and all sorts of avenues opened up for sparking enthusiasm whilst aiding understanding. The first of Gardner's books to be published by Pelican was *Mathematical Puzzles and Diversions* and I've just reached for my repeatedly-read and hence rather battered copy to remind myself about polyominoes.

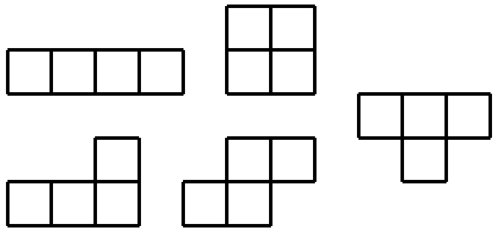
Polyominoes entered this world in a talk given by a 22-year old student, Solomon Golomb, to the Harvard Mathematics Club in 1953. True, there were some anticipations, including a pentomino problem published in 1907 in Henry Dudeney's *Canterbury Puzzles*. But Golomb introduced generality and extension. His early results were brought together in 1965 in a book published in the States, *Polyominoes: Puzzles, Patterns, Problems, and Packings*, a version of which was printed in Britain the next year. By then, polyominoes were also being featured by Martin Gardner in his 'Mathematical Games' column in *Scientific American*.

What are polyominoes?

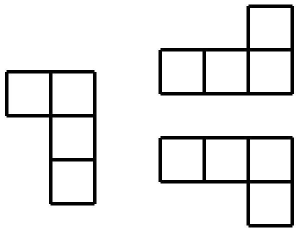
We are familiar with the shape known as a *domino*; it consists of two identical squares stuck together side to side. The family of shapes created by sticking together many squares gets the name *polyominoes* by extension. From the family name back down to the designs made specifically from 3, 4, 5, 6 … squares we have the *tromino*, *tetromino*, *pentomino*, *hexomino* and so on. Clearly, there is only one domino but two different trominoes,



and five tetrominoes, the basis for the video game *Tetris*.



It may be important to have a discussion about why we do not count as distinct these L-shaped tetrominoes.

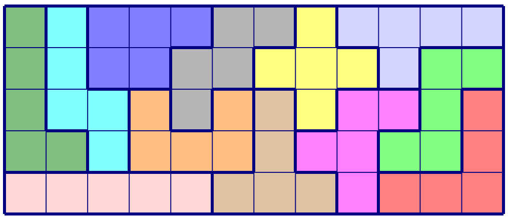


We are already accumulating questions to ask:

1. Is there a way of making the tetrominoes from the trominoes? *By accretion. Simply stick an extra square on to the trominoes in a variety of positions.* By adding a square to the straight tromino, which of the tetrominoes can be made? By adding a square to the right tromino (L-shaped tromino), which of the tetrominoes can be made?
2. Imagine the designs are made from matchsticks. Are both trominoes made from the same number of matchsticks and are the tetrominoes made from the same number of matchsticks? *Yes, 10; No, 13 each time except for the square, 12.*
3. What symmetry features does each design have? Which tetrominoes have reflective symmetry and for those that do, how many axes are there? Which have rotational symmetry. Clearly, we are into standard curricular content and skills via a non-standard route.

Activities and problems

Finding all the pentominoes is a lovely task for individuals to tackle in the classroom, and finding all the hexominoes is a great whole-class or group activity. There are twelve of the former and they fit nicely into a 5×12 rectangle:



It's not the only way to fit them in — there are over a thousand distinct solutions.

Task 1

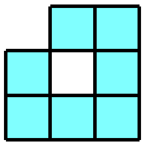
Find ways of fitting the twelve pieces into a

- 4×15 rectangle
- 6×10 rectangle.

There are 35 different hexominoes but beyond that the numbers grow exponentially, roughly by a factor of 4 each time and hence beyond any classroom use.

n	1	2	3	4	5	6	7	8
$P(n)$	1	1	2	5	12	35	108	369

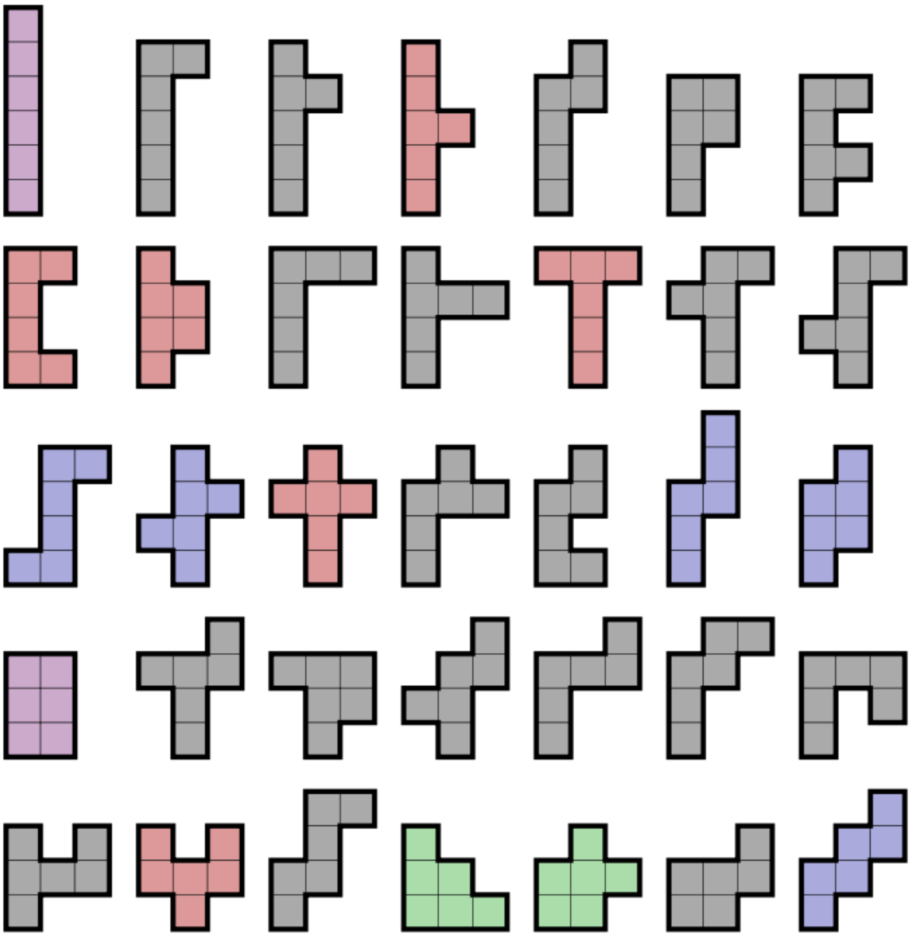
In any case, problems start to arise as to what designs to include and what to ban. For example, should the wrap-around heptomino with a 'hole' be allowed?



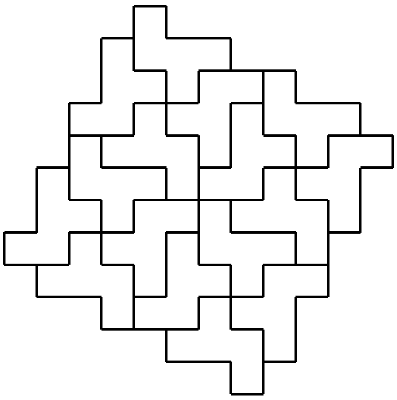
At the bottom of the page are the 35 hexominoes (in an image from Wikipedia: R A Nonenmacher, CC BY-SA 4.0).

Task 2

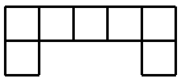
Identify the hexominoes that can be folded to make a cube (i.e. find the nets of a cube).



Armed with a multitude of polyominoes (up to the hexominoes), a natural question to ask is whether they can each be used to tile the plane (tessellate). This pentomino can:

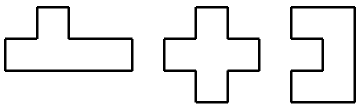


In fact, no issues arise until the heptominoes and then several cannot be used for tiling the plane including

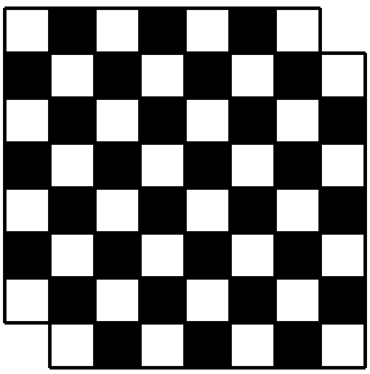


Task 3

Show how each of the following pentominoes tessellate.



A classic problem included in Golomb's book but actually unveiled in 1946 in Max Black's *Critical Thinking* is the mutilated chessboard problem. A chessboard has had opposite corner squares snipped off, so that it looks like this:



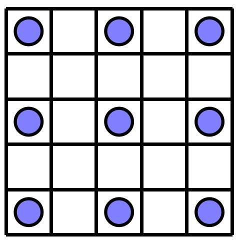
Now take a pile of dominoes each of which covers two adjacent squares. Can they be placed on the mutilated board so that it is completely covered? In fact in can't and the proof is rather lovely. The mutilated board has 62 squares, so ostensibly 31 dominoes are required. But of the 62 squares, 32 are white but only 30 are black. Since each domino must cover one white and one black square, the number of white and black squares covered must be equal. So the task is impossible.

Task 4

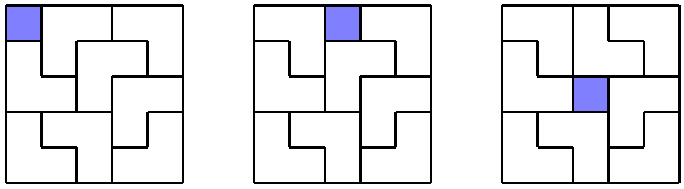
Show how the twelve pentominoes can be placed on a mutilated chessboard with all four corner squares missing.

Twenty years ago, I was editing *The Changing Shape of Geometry*, a centenary celebration of the founding of the Mathematical Association's Teaching Committee. I got in touch with both Martin Gardner and Solomon Golomb and, rather speculatively, asked them to contribute to it. Gardner had just lost his wife and, quite understandably, the idea of writing for a teacher's association across the pond was not uppermost in his thoughts but Golomb responded enthusiastically. Following his talk on polyominoes back in 1953 he had gone on to complete a doctorate at Harvard on the distribution of prime numbers, and enjoyed a stellar career at Caltech and the University of Southern California, becoming a leading authority on pseudorandom numbers (used in mobile phones) and much else. And yet, he was still nurturing his polyominoes like a mother hen, 'irrevocably committed to their care and feeding', as he put it. There is a result, he explained to me in 2001, that he had used in talks but which has never been published, and he would be happy for me to use it in a Mathematical Association book. Here is the result and proof in Golomb's own words.

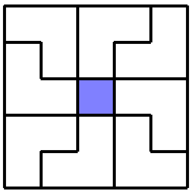
> ***Problem*** Where can a single square be removed from a 5×5 board so that the remaining 24 squares can be tiled by 8 right trominoes?
>
> ***Answer*** If and only if the single square is removed from one of the 9 marked locations.



> ***Proof*** First, observe that no right tromino can cover more than one of the 9 dots. If no dotted square is removed, at least 9 right trominoes will be required to cover the rest, but 9 right trominoes have a combined area of 27, whereas the entire 5×5 board has an area of only 25. If a dotted square is removed, there are only three distinct cases to consider:



I wonder if Golomb should have given the last possibility in the symmetrical form

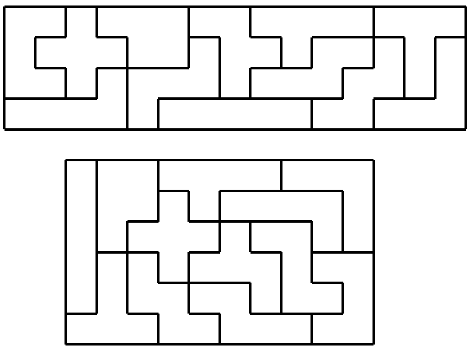


and whether it is possible to find alternative versions of the first two to include some element of symmetry. Something to explore in class, perhaps?

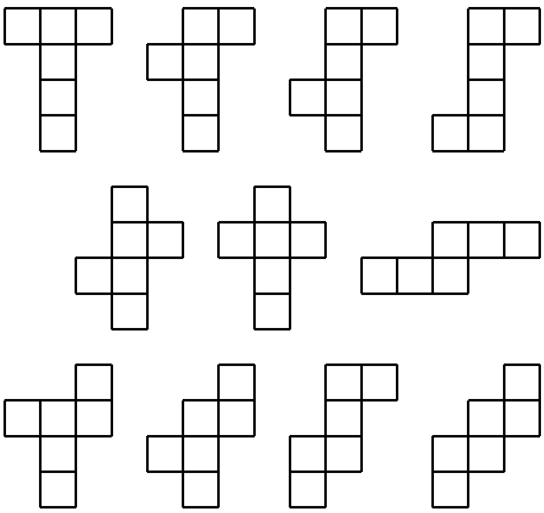
Incidentally, there is a splendid appreciation of Golomb's life and work written by Stephen Wolfram (see references below).

Sample solutions to tasks

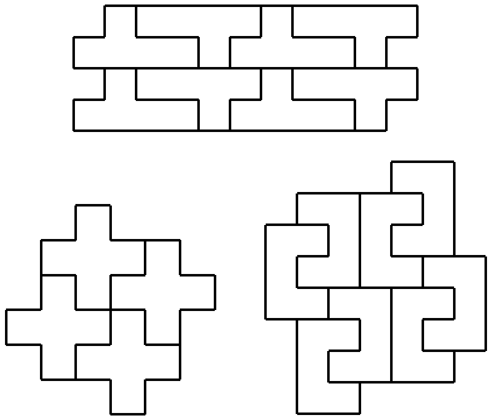
Task 1



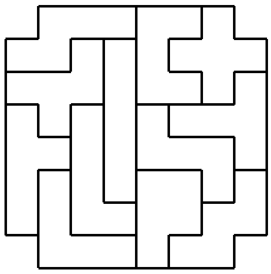
Task 2



Task 3



Task 4



References

Martin Gardner, *Mathematical Puzzles and Diversions*, Pelican, 1965; especially Chapter 13.
Solomon W. Golomb, *Polyominoes*, George Allen & Unwin, 1966.
Jan Kok, 'Polyominoes: Theme and variations': www.cs.brandeis.edu/~storer/JimPuzzles/PACK/Pentominoes/LINKS/PentominoesJankok.pdf (posted on 14 August 2007).
Chris Pritchard (ed.), *The Changing Shape of Geometry*, CUP/MAA, 2003, pp. 343-345.
Stephen Wolfram, 'Solomon Golomb (1932–2016)', posted online (25 May 2016) at https://writings.stephenwolfram.com/2016/05/solomon-golomb-19322016/