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Stagecoach: The Influential Zoltán Diénès

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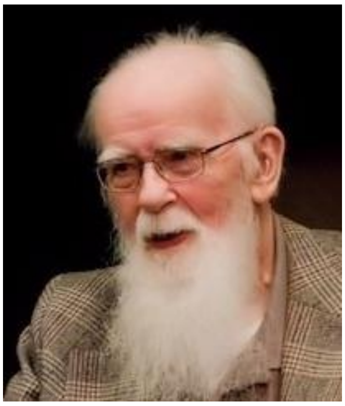
Most primary teachers and some secondary teachers will be familiar with the objects shown in the picture below and will have made use of them, perhaps extensively, in their lessons.

Multibase Arithmetic Blocks (MAB, here base 10) often carrying the name Dienes, after their inventor, consist of differently coloured manipulatives representing units, tens, hundreds and thousands, the last three being dubbed *longs*, *flats* and *blocks*. They enable young children to develop number concepts, particularly place value.

Back in 2016 I was fortunate enough to attend ICME-13 in Hamburg. I was there as a secondary mathematics specialist but learnt more about the primary side than I expected. And I was taken with one session in particular, a session by Mike Thomas of the University of Auckland, a long-standing appreciator of Zoltan Diénès as one of the leading figures in the application of psychology to mathematics learning, especially (but not exclusively) in the early years. The two had had contact towards the end of Diénès' life (he died in 2014 at the age of 97) and Diénès asked that the Hungarian diacritical marks be restored to his name in any work that Thomas produced, so that the correct pronunciation (Diaynish) might be reestablished. But to this day, we continue to mispronounce his name when we say 'Deans'.

Born in Budapest during the First World War, Zoltán Pál Diénès was raised by parents with a strong mathematical background. His early experiences included a Montessori education, much moving around from country to country, including living in a commune in Nice, picking up languages on the way, and eventually washing up in England at the age of fifteen. Diénès was subsequently awarded a BA degree from the University of London in 1937 and defended his doctoral thesis there two years later. With a thesis title of 'Constructivist foundations according to Borel and Brouwer' we may conclude that Diénès was no mathematical slouch. He initially turned to teaching, first at Highgate School in London and then at Dartington Hall School in Devon, renowned for its progressive education with a minimal focus on formal classroom activities.



Zoltán Diénès

Diénès then turned to psychology, studying the learning of a 10-year old and started to build the new discipline of psychomathematics. His academic career developed at four English universities, after which he held more senior positions at Harvard and Adelaide, before becoming Director of the Psychomathematics Research Centre at Sherbrooke University in Quebec. Both during his tenure and afterwards, Diénès toured the globe researching and explaining his theories.

Here is what Diénès had to say about his motivation for studying psychomathematics and in particular about that first study, conducted whilst a Lecturer in Mathematics at the University of Leicester:

> As I became interested in the problem of why mathematics was found difficult by most people, I wondered whether the difficulties in the foundations of mathematics had not

something to do with the difficulties that children experienced in understanding mathematics.

I chose the age of ten as a reasonable stable period of childhood and ran a concept formation experiment on a representative sample … I tried, in a small way, to reorganise mathematical work in some classrooms, turning the classrooms into laboratories of discovery and construction, using specially designed materials which later developed into the now well-known multi-base and algebraic materials. It was impossible to contain the experiment as a psychological one because of its immediate and unqualified success. The experiment turned into a mathematics project throughout the County of Leicester. During this time, 1958-59, I worked out, through practical ways, some principles which should account for the success observed.

The findings of this research were written up and published in papers published in 1959, 'The formation of mathematical concepts in children through experience' and 'The growth of mathematical concepts in children through experience'.

Let us wind back a little. The accepted learning theory in mathematics in the early part of the twentieth century focussed on teacher exposition and learner absorption using choice examples followed by graded problems. Computational fluency, often gained through hard grind was followed by a procedural approach to algebra. There was little attempt to understand how learning occurs and hence what should happen in the mathematics classroom to develop deep understanding.

This was to change in the 1950s with the emergence of theories of learning mathematics as a process of building up mental representations, developing skills in using and modifying them. Something emerges from inside the child rather than being acquired through osmosis. The name most recognisable in this context is Jean Piaget, about whose ideas a large body of theory has been generated, perhaps followed by Jerome Bruner.

Diénès' theories were very much based in practice. They were certainly more practical than the theories of Piaget and Bruner, more typically prescriptive rather than descriptive. Diénès did take the 'reflection abstraction' of Piaget, and in fact in 1967 he argued that we abstract in cycles, moving upwards as we go. If we reach abstraction and see that it applies elsewhere then we have generalisation. For example, we abstract to classify counting numbers as odds or evens, then generalise to get odd + odd = even, etc.

Diénès' Principles (1960):

- ***Dynamic Principle***
  Preliminary, structured activities using concrete materials should be provided to give necessary experiences from which mathematical concepts can be built eventually. Later on, mental activities can be used in the same way.
- ***Constructivity Principle***
  In structuring activities, construction of concepts should always precede analysis.
- ***Mathematical Variability Principle***
  Concepts involving variables should be learnt by experiences involving the largest possible number of variables. So to understand what constitutes a parallelogram we need to vary lengths, angles, orientation and so on.
- ***Perceptual Variability Principle***
  In order to allow as much scope as possible for individual variations in concept-formation and to induce children to gather the mathematical essence of an abstraction, the same conceptual structure should be presented in the form of as many perceptual equivalents as possible.

Diénès' Theory of Learning Mathematics (1973):

Essentially there are six stages, which have been summarised by Melanie Clouthier at the Zoltán Diénès website. Here is that summary pared down even further:

free play, including 'trial and improvement',

playing by the rules, including recognising the advantages of rules,

comparison, i.e. comparing games, basic analogy, common structures in games and activities,

representation, as a way of picturing the abstract concepts emerging in the mind of the learner,

symbolisation, i.e. studying the representation with a view to catching some elements in symbolic form,

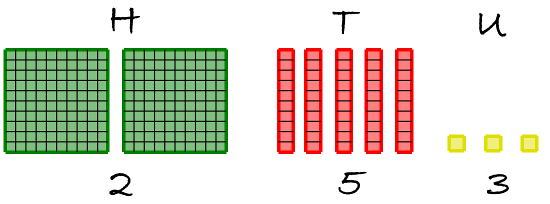
formalisation: providing structural elements such as axioms and a framework for proofs.

And central to all of this is the role of manipulatives. A key concept of Diénès was that of 'embodiment'; this involves concrete or pictorial objects combined with physical action, all leading to abstraction and generalisation. As we've seen, he used multibase blocks (not just base 10 blocks). They are an excellent way to teach place value but prove amazingly flexible.

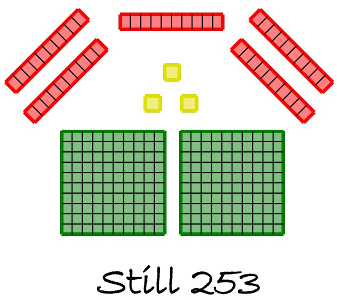
The flexibility of Dienes' blocks

1. Place value

Taking two flats (2×100), five longs (5×10) and three units (3×1), we can lay them out like this to represent 253.



We could move the manipulatives around without removing any or adding to them to demonstrate conservation of number.

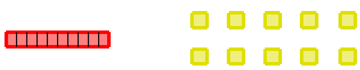


Perhaps where Dienes blocks have the edge over the Empty Number Line (ENL) is in giving the impression of relative size. There is a sense of proportionality that is simply missing from the ENL and perhaps this is one reason why the ENL has proved less effective when used with weaker pupils than with more able pupils.

2. Exchange in addition and subtraction

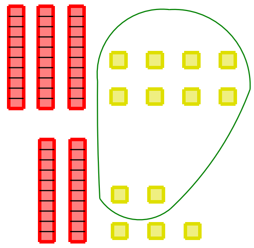
The concept of exchange is important in many walks of life, from "I'll swap my Teenage Mutant Ninja Turtles cards for your UEFA Champions League cards" to exchanging Som for Tenge at Chernayevka–Zhibek-Joly. (Year 4 pupils will help you understand the first, whilst a Google search is almost a necessity with the second!)

So if a long and ten units are equivalent



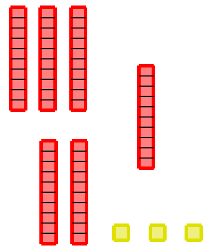
it is perfectly reasonable to make an exchange.

When adding, $38 + 25$ may be shown as



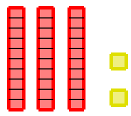
$$(3 \times 10 + 8 \times 1) + (2 \times 10 + 5 \times 1)$$
$$= 5 \times 10 + 13 \times 1$$

or, upon exchanging ten units for a flat, as

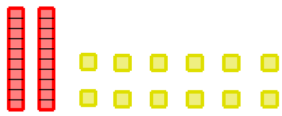


$$6 \times 10 + 3 \times 1.$$

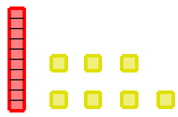
Similarly, in subtraction, if we wish to subtract 15 from 32, we would start with



Then by exchange one of the longs for ten units



it is possible to remove a long and 5 units to leave

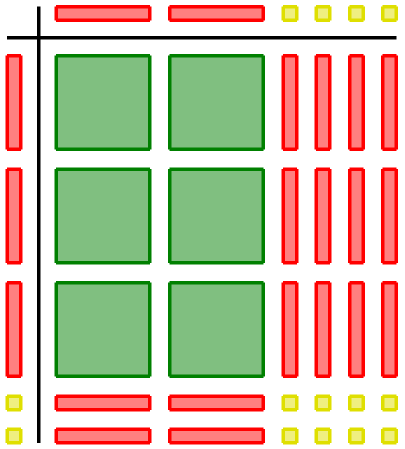


What we have done is:

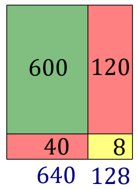
$$\begin{aligned} 32 - 15 &= (3 \times 10 + 2 \times 1) - (1 \times 10 + 5 \times 1) \\ &= (2 \times 10 + 12 \times 1) - (1 \times 10 + 5 \times 1) \\ &= 1 \times 10 + 7 \times 1 \\ &= 17. \end{aligned}$$

3. Multiplication and beyond

As complexity increases, and let's face it multiplication is more complex than addition and subtraction, using Dienes representations (or handling the manipulatives) becomes fiddlier. The diagrams can be made a little simpler by taking out the grid lines and simpler still by moving from an array to an area diagram.



This array to the left shows 32 down the left-hand side and 24 across the top. In effect there are four partial sums (600, 40, 120, 8), which can be paired (640, 128) to give final answer 768.

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   32
 × 24
  128
  640
  768
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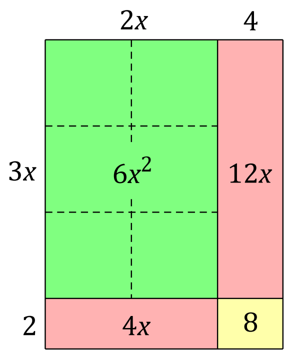
But exactly the same moves are being made when we use the multiplication algorithm. In other words, the array bolsters learners' conceptual understanding of how the algorithm works, stripping away the drabness of the algorithm when used mechanically and without context.

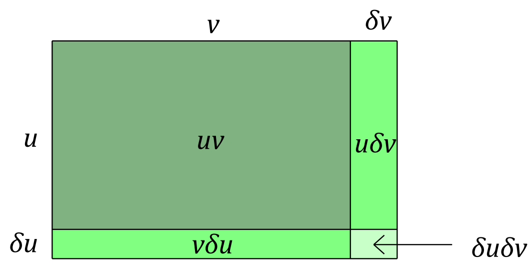
To the secondary teacher, much the same diagram can be used to model the multiplication of linear terms (pairs of brackets, if you like) to give the quadratic product.



$$(3x+2)(2x+4) = 6x^2 + 4x + 12x + 8$$
$$= 6x^2 + 16x + 8.$$

For a much more detailed and extensive treatment, see 'Visualizing Algebra', Chapter 5 of my book, *Pack up a Penguin*. Included there are the distributive law, factorising quadratics, difference of squares, completing the square and the square root algorithm, as well as a stab at handling cubics through volume diagrams.

The area model works well too in calculus, for example in demonstrating the product rule for differentiation.



In my depiction, colour is used to attempt to show that $\delta u \delta v$ 'pales into insignificance' in comparison with not only uv but also $u\delta v$ and $v\delta u$. If this small quantity is neglected, the standard result follows directly.

We may think of some of these ideas as latter-day extensions of Diénès' ideas, but he himself did not stop at arithmetic. As we have here, he saw possibilities for the use of manipulatives in algebra and beyond. His Algebra Blocks, described in *Building up Mathematics* (1960), were easier to manipulate than algebraic symbols, in his view. His Logic Blocks with attributes (colour, size, shape) and an associated binary representation, were used in an additive and multiplicative fashion, essentially a step towards the concept of a field. Finally, he found a way to embody vectors. We can only marvel at the sort of imagination Diénès must have had to create such an array of physical and pictorial objects to encapsulate the essences of mathematics and provide opportunities to bolster conceptual understanding.

Further reading

For a survey of Diénès' work try Googling names such as Merlyn Behr, Tom Post, Dick Lesh, Elena Nardi, Aiso Heinze and John Monaghan. For a useful summary of Diénès' ideas, read Anthony Orton, *Learning Mathematics: Issues, Theory and Classroom Practice*, Continuum, 3rd Edition, 2004 (especially from page 176). I particularly like a paper by Serigne Mbaye Gningue, 'Remembering Zoltan Dienes, a Maverick of Mathematics Teaching and Learning: Applying the Variability Principles to Teach Algebra', *International Journal for Mathematics Teaching and Learning* 17, 2 (2016), available via www.cimt.org.uk/ijmtl/index.php/IJMTL/article/download/17/13.

For a feel for what Diénès' stages might look like, there are four articles from him in Volumes 29 (2000) and 30 (2001) of *Mathematics in School*. The first of them suggests the stages for those developing an understanding of basic whole number work. All the stages, including the convoluted fifth and sixth stages, are detailed in the article in Vol. 29, no. 2 (March 2000), pp 27-32. Finally, Allan Duncan looks at Dienes blocks and the Empty Number Line in 'Should we bin our Dienes?', *Primary Mathematics* 10, 1 (Spring 2006), 18-22.