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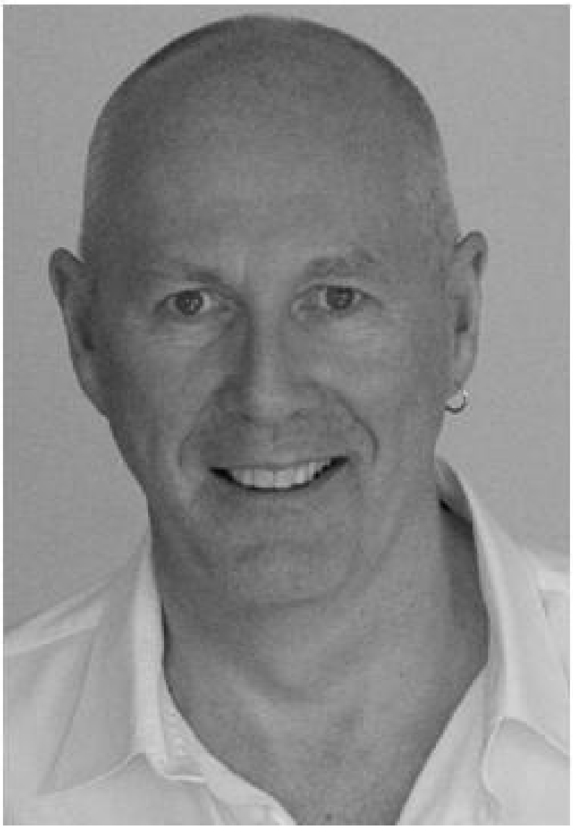

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Reasoning as a mathematical habit of mind

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Introduction

In this paper I look at different aspects of mathematical reasoning, and argue that we need to make sure students of all ages engage in a range of mathematical reasoning, particularly given the evidence that teaching for reasoning is a powerful complement to teaching that is more focused on skills and procedures.

A major report [1] published in the USA put forward a model of mathematical competency as comprising five strands woven together (this metaphor deliberately chosen on the basis of a rope being stronger than the sum of its strands): procedural fluency, strategic competence, adaptive reasoning, conceptual understanding and productive disposition. Since the publication of this report, there has been growing consensus, within the world of mathematics education, on the importance of at least some, if not all of these strands. Linked to this, England's 2015 National Curriculum for Primary Schools [2] has the stated aim of developing fluency, problem solving and reasoning and the language of these three proficiencies is increasingly also being used when talking about the curriculum for Secondary Schools.

Curriculum supporting documents often present these proficiencies in the order in which they are stated in the curriculum document and that, along with the popularly held view that fluency in basic arithmetic is needed before students can engage in mathematical reasoning, means that mathematical reasoning is sometimes talked about, and enacted in classrooms, as the icing on the cake of mathematics teaching and learning – something a few students (the mathematically talented ones) get to engage with. A result of this is that some pupils then experience fewer opportunities to engage in mathematical reasoning than their peers. I argue here that an emphasis on mathematical reasoning is an educational right that all students are entitled to, that reasoning is complementary to procedural fluency, not an outgrowth of it, and that rather than being some esoteric way of thinking that only a minority of students can engage in, mathematical reasoning is achievable by the vast majority of students. In making reasoning available and accessible to most students, not only will they deepen their mathematical understanding, they may also get a stronger sense of the pleasure of mathematics, of the romance of mathematics, to paraphrase the words of the illustrious previous president of the Mathematical Association, Alfred North Whitehead in [3].

Reasoning or arithmetic?

Before reading on, I invite you to consider whether each of the following statements is true or false.

$$39 \times 46 = 39 \times 45 + 46$$

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The mathematically astute reader (as the reader of this journal doubtless is!) may invoke the associative law, reasoning that $39 \times 46 = 39 \times (45 + 1)$ leading to $39 \times 45 + 39$, thus establishing the second statement as true (and consequently that the first statement is false). Offering these equations to teachers and students, given that the numbers are sufficiently ‘ugly’ to discourage checking by calculating each side of the equation, a conversation typically ensues as to whether to ‘read’ 39×46 as ‘forty-six groups of thirty-nine’ or as ‘thirty-nine groups of forty-six’, thus engaging participants in some reasoning about the commutative nature of multiplication and when it is useful to invoke its use. The reading of ‘forty-six groups of thirty-nine’ can lead to the awareness that if this is reduced to ‘forty-five groups of thirty-nine’ (39×45), then to preserve the equality, another group of thirty-nine needs to be added, hence the second equation is true.

Notice that in each case, reasoning as to whether each statement is true or false is independent of any ability to carry out any of the calculations presented. Now consider the second equation within a string of similar examples:

$$3 \times 5 = 3 \times 4 + 3$$

$$39 \times 46 = 39 \times 45 + 39$$

$$328 \times 18 = 326 \times 17 + 326$$

The reasoning required to establish that the first equation is true is no different from that required to establish the truth of the second or third statements. The last equation is adapted from an item on one of England's recent national tests for the end of primary school. At the time of writing, students, on exit from primary school, sit three mathematics tests − one arithmetic test and two ‘reasoning’ tests. The last question on the 2016 second reasoning test gave the equation:

$$5542 \div 17 = 326$$

Students were asked to show how they could use this equation to find the answer to 326×18, in other words, to apply the reasoning implied in the third example in the string above. This was the most poorly answered question of all the questions across all three papers, with only 26% of pupils answering it correctly. Given that this was the final question across all three papers, the test setters presumably thought it was the hardest item on the test: the low success rate would appear to support this. One might expect that the low score was a consequence of many students simply not getting that far on the paper, but 79% actually attempted an answer. I suspect that many of these tried to carry out some calculation, but, if one can engage in the sort of ‘reading’ and reasoning outlined above, then the only real challenge that the problem presents is appreciating that $5542 \div 17 = 326$ informs you that $326 \times 17 = 5542$, and then to reason that 326×18 is must therefore be $5542 + 326$.

This example gets at the heart of the distinction in [4, p. 3] that the psychologists Nunes, Bryant, Sylva and Barros make between arithmetic ('learning how to do sums and using this knowledge to solve problems') and mathematical reasoning ('learning to reason about the underlying relations in mathematical problems they have to solve'). From a number of research studies with young children Nunes and colleagues argue that being able to do arithmetic and being able to reason mathematically cannot be treated as proxies for each other and that mathematical reasoning needs to be attended to in its own right. They examined this by tracking students in a five-year long longitudinal study, concluding that reasoning and arithmetical abilities contribute independently to predicting progress in learning mathematics, but that of the two 'mathematical reasoning was by far the stronger predictor' [5, p. 136] of later success and that teaching must address improving reasoning skills as well as, and separately to, arithmetical skills.

A language of mathematical reasoning

If mathematical reasoning is to play a more central and distinct role in teaching and learning, then I think it helps to bring some clarity to what it might look like in classrooms. A search across the mathematics education literature brings up a range of terms involving reasoning, including additive and multiplicative reasoning, statistical reasoning, proportional reasoning, approximate reasoning, geometric reasoning, both positional and axiomatic and so on.

More concisely, [2, p. 99] describes the aim of being able to reason in the following terms:

> reason mathematically by following a line of enquiry, conjecturing relationships and generalisations, and developing an argument, justification or proof using mathematical language.

Working with teachers I am often asked if reasoning is not simply part of problem solving, that, if students are working on solving problems (genuine ones, that is problems to which they do not have an immediate solution method), then surely that must involve some form of reasoning. I would agree, but reasoning needs to go beyond finding an answer to a particular problem. It needs, as in the NC definition above, to move to thinking about generality, and so problem solving is embedded within reasoning. Reasoning is broader than finding a solution to a particular problem, since reasoning is driven by the desire to ask 'what is the general mathematical structure that this particular problem is only a single instantiation of?'

If we take seeking generality as the core of mathematical reasoning then I find it helpful to attend to different types of reasoning. Two of these, deductive and inductive reasoning, are part of the canon of mathematics and need little introduction. But three others, abductive, analogical and relational reasoning, are perhaps less often acknowledged but are important in teaching and learning mathematics. Let us look at each of type.

Deductive reasoning

Magic squares are a classic example of deductive reasoning: given the sum of the rows, columns and diagonals, and some cell entries given, can the other cells be completed? In [6], Arcavi developed an extension to the traditional magic square by allowing numbers to be repeated. So, given that a number can be used more than once, which of the squares in Figure 1 can form a magic square?

	3	
2		1

Sum of 9

	2	
1		5

Sum of 6

	4	
2		2

Sum of 8

FIGURE 1: Which of these can be completed to make a magic square?

The second example introduces the use of negative number, while the third example raises the possibility of a solution not always being possible, opening up inquiry into the necessary conditions for being able to complete a square.

Inductive reasoning

Take these four calculations

$$5 \times 6 =$$
$$7 \times 8 =$$
$$4 \times 5 =$$
$$6 \times 7 =$$

Presented thus, they may provide students with an opportunity to practise recall of multiplication bonds, but offer little opportunity for mathematical engagement beyond that.

Now consider the same calculations (and their answers) in a different order:

$$4 \times 5 = 20$$
$$5 \times 6 = 30$$
$$6 \times 7 = 42$$
$$7 \times 8 = 56$$

Asking learners, what do you notice? and What do you wonder? can lead to them noticing that the answers increase by differences of 10, 12, 14, and to wondering if this pattern will continue. By simply re-ordering the calculations, inductive reasoning is invoked (in the more everyday sense of pattern spotting, than in the sense of proof by induction) raising a mathematical awareness that could be explored further.

Abductive reasoning

Largest-smallest difference is a classic example of a task that invokes abductive reasoning - noticing a repeated connection. Close to, but not quite the same, as inductive reasoning, abductive reasoning is based in noticing similarities and co-occurrences of phenomena.

Write down three digits

1. Arrange the digits from largest to smallest as a single number
2. Arrange the digits from smallest to largest as a single number
3. Find the difference between the two numbers.

Repeat with the answer

What do you notice? What do you wonder?

Here we see the complementary relationship between abductive and deductive reasoning. Trying out several examples raises the possibility that there is a generality, but establishing whether or not the generality will always hold is established by deductive reasoning. It is important that even early in their experiences of learning mathematics students get exposed to the idea that, when asked, 'Will that always work?', that the answer 'Well it did the five times I tried it' is not mathematically sufficient.

Analogical reasoning

A mathematical answer to 27 divided by 6 is 4 remainder 3.

Before reading on, I invite you to think of a simple real-world situation that can be modelled by dividing 27 by 6 but where it makes sense, back in the real world, to round the answer up to 5.

Asked to do this, people come up with situations comparable to 27 people going somewhere, booking taxis that can carry 6, and so needing to book 5 taxis, or packing all of 27 eggs into boxes of 6. Whatever the situation it is most likely that the context chosen is some sort of 'packing' situation where one 'container' is not completely full, resulting in the need to round up the mathematical answer.

Research has shown that expert problem-solvers do not always treat new problems from scratch, but, instead, 'match' the problem analogously to other similar problems that provide a solution image or approach [7]. Here, the archetypal divide-and-round-up-the-answer analogy is to a 'packing' problem. There is an extensive body of research into the power of working with 'core' archetypal problems for developing understanding of additive and multiplicative structures, and I recommend anyone who is interested to read [8].

Relational reasoning

Consider the situation below (adapted from [9]).

- On Saturday some friends came to tea. We shared a packet of biscuits, equally.

- On Sunday I had another tea party.
- A greater number of friends came around on Sunday.
- We shared, again equally, the same number of biscuits as there were on Saturday.

Did each person on Sunday eat more, less or the same as each did on Saturday?

Even very young children can reason that everyone on Sunday gets less to eat (working on the assumption that these people exist in a mathematical universe where no one is on a diet, gluten intolerant or gives up their share!). Given that there are three possible situations for the number of friends on Sunday (fewer, same, more) and for the number of biscuits (fewer, same, more), then there are nine possible differences between Saturday and Sunday, and in seven of these it is unambiguous as to how the Sunday situation compares to Saturday.

As this example shows, reasoning about relations between quantities - relational reasoning - can be done without needing to know the actual numerical values of the quantities. Indeed, arithmetical problem-solving requires the relationship between quantities to be figured out before actually working on the calculation. Attending too quickly to the actual quantities, rather than the relationship between then, can lead to erroneous reasoning, as this example shows:

> Two battery-operated cars - one red, one blue - are travelling equally fast around a track.
>
> The red car started first.
>
> When the red car had completed 9 laps, the blue car had completed 3 laps.
>
> When blue car had completed 15 laps, how many laps had the red car completed?

Offered this problem, many students foreclose quickly to the answer of 45 laps - the various ratios between the values given make this a seductive answer – 3 : 9 as 15 : 45 or 3 : 15 as 9 : 45. The correct answer of 21, given that the cars are travelling at the same speed, and so red is always 6 laps ahead of blue, cannot be established from examining the numbers, only from the information in the first sentence (often glossed over by the reader).

The absence of reasoning in classrooms

Despite the evidence both for the importance of students engaging in mathematical reasoning, and for its importance in later development, much evidence points to a paucity of reasoning in mathematics lessons. Why might this be so? It is beyond the scope of this paper to explore in detail the variety of reasons for the lack of attention to reasoning (not the least of which is the climate of high-stakes testing to which schools are subjected), but two possible constraints are worth touching upon: the emotional cost of reasoning and the focus of planning.

A major research project in the USA [10] worked with a group of upper primary and middle school teachers to design and implement a number of lessons that would have mathematical reasoning at their heart. The teachers came together to discuss the mathematical tasks forming the core of the lessons, to work through the tasks and to come to a common understanding of the purpose of the lessons, which was to promote reasoning. In all, 68 lessons were subsequently enacted in classrooms and the research team visited and observed all these lessons. The main finding from the research was that shortly after the start of each lesson, over two-thirds of the lessons quickly 'declined' (the researchers' term) into lessons that either simply involved students carrying out routine procedures (as a result of the teacher telling them what to do), working in ways that were non-systematic and even, in some cases, students engaging in non-mathematical activity (for example, colouring in diagrams).

Looking in more detail at the third of lessons where mathematical reasoning was maintained, the researchers identified a number of factors common across these lessons, factors that included building on students' prior knowledge and providing an appropriate amount of time (too much being as unproductive as too little). One notable factor for maintaining a focus on reasoning was evidence of 'sustained pressure for explanation on meaning'. Now teachers often, with good intent, want to make learning as pleasurable as possible, so 'sustained pressure' sounds like the antithesis of this, but the work of the Nobel prize winner, Daniel Kahneman, points to why such pressure might be necessary.

Kahneman proposes a model of our thinking as being either 'fast' or 'slow' [11]. Fast thinking is that which we do without having to reflect on it - knowing that seven times eight is fifty-six or that tan is sine divided by cosine. Slow thinking is more deliberate; it is the kind of thinking we have to engage in when doing mathematical reasoning. Over a number of studies, Kahneman has shown that moving from fast to slow thinking is often accompanied by a slight, but noticeable feeling of depression. The significance of this to teaching is that teachers recognise that moment when having given a class a challenging task, they can feel the energy in the room changing, and not in a positive direction. It seems reasonable to assume that one result of many of the lessons in Henningsen and Stein's research declining into routine procedures was a result of the negative energy arising from students moving into 'slow' thinking with some teachers 'easing' the classroom climate by pointing out what to do, whilst those teachers applying 'sustained pressure' helped students to work through their resistance.

A second possible reason for limited attention to mathematical reasoning in lessons may arise from what teachers attend to in planning. The researcher Ference Marton argues that in any teaching and learning encounter, there are 'objects' of learning coming into being, and that these can either be direct objects or indirect objects [12]. Direct objects of learning are those aspects of lessons that students are immediately attending to. They are what students might say in response to the question 'what did you do in maths today?' - 'We worked on subtraction', 'We solved simultaneous equations', and so on. Marton argues that, whether intentional or not, every direct object of learning invokes a number of indirect objects of learning. For example, the student working through a page of simultaneous equations but without much sense of where such equations come from, or what the results mean, may come to learn that mathematics comprises a number of procedures that simply have to be committed to memory. That learning may not have been the direct intention of the teacher, but is an indirect consequence of the direct object of learning worked on.

In my experience, teachers, when planning mathematics lessons, often focus mainly on the direct object of learning - fractions, money, linear graphs and so on. Tasks are then collected together that address the direct object. While that might be the starting for planning, it is also important to ask what the indirect outcomes might be of working on those tasks - what mathematical activity students are going to be engaging in as a result of working on the tasks? This is important in promoting mathematical reasoning, as reasoning cannot be taught directly. The direct task given to students is only a starting point. Having chosen a task, we have to think our way into what students are likely to do as a result of engaging with it, that is, what indirect learning might come about through working on the direct object. I recently visited Japan to observe not only a number of Lesson Study meetings (a form of professional development for a teacher where they observe and critique a carefully crafted lesson) but also a number of regular lessons. In the discussion following each of the lessons, a common focus was on what had been anticipated that the students might do mathematically, and whether or not this came about in the course of the lesson. Anticipating and working with the mathematical reasoning of the students was treated as more important that thinking about what the teacher had to do in the lesson.

Reasoning as a habit of mind.

Another possible reason for why reasoning may get less attention in lessons is the perception that is has to be the focus of an entire lesson, and given the amount of 'content' to cover, this may only be able to happen occasionally. In [13], Cuoco, Goldenburg and Mark describe mathematical power as a 'habit of mind', so if we want students to reason mathematically then this needs to become a habit of mind, and, like any habit, is best developed little and often. Rather than the occasional 'inquiry' lesson, reasoning chains – a series of linked, short, activities – can engender such a habit of mind.

We looked at such a reasoning chain earlier in discussing whether or not $326 \times 18 = 326 \times 17 + 326$ was true: the two simpler but structurally identical examples preceding this would move the conversation with students away from thinking about calculating answers to discussing and reasoning about the underlying structure. That example was structured around preserving the underlying structure whilst making the numbers involved appear to be more challenging. Another approach is to present a series, a chain, of calculations, where subsequent answers can be reasoned about based on the previous example. For instance:

$$160 \div 16$$

$$320 \div 16$$

$$320 \div 32$$

The reasoning here could go along the lines of, given that the answer to the first calculation is easy to calculate mentally, then if the dividend is doubled but the divisor unchanged, then the answer is going to double. The third example provides an opportunity to discuss why doubling both the dividend and divisor leaves the answer unchanged, and exploring why this is not the case with multiplication. Cathy Fosnot has written about such chains, and I recommend her work if you are interested to read more (see, for example, [14]).

Conclusion

If reasoning is going to become more central to most mathematics teaching and learning, then there are some shifts that need to come about in lessons. The most important shift is to move away from thinking that getting answers to problems is the end goal of the lesson. Answers have to be seen not as the end product of a lesson, but as the beginning, as an opportunity to examine the underlying mathematical generality. That involves a shift from asking:

> How to I teach students to answer this problem?

to

> What mathematical reasoning do I expect them to engage in as a result of working on this problem?

Currently the business of education with its focus on test results may not be conducive to such shifts, but we should not lose sight of the broader aims of education, aims that were important for John Dewey, over 80 years ago, and, in our internet age, may be even more important now:

'While it is not the business of education ... to teach every possible item of information, it is its business to cultivate deep-seated and effective habits of discriminating tested beliefs from mere assertions, guesses, and opinions.' [15].

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