102.08 A method of evaluating $\zeta(2)$ and $\int_0^\infty \frac{\sin x}{x}\,dx$

Introduction

I have recently been involved in setting up a new DfE-funded resource aimed at students preparing for STEP* [1]. I came across an interesting question while trawling through the STEP data base [2] for suitable material. It is from 1998 and I am surprised that I failed to notice at the time that only a few more calculations are needed to provide a rather simple evaluation of $\zeta(2)$, expressed as an infinite sum:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}. \tag{1}$$

The calculation involves the integral

$$\int_0^\infty \frac{\sin x}{x}\,dx, \tag{2}$$

the value of which $\left(\frac{1}{2}\pi\right)$ is well-known; however, it was a pleasant surprise† to find that it can be evaluated by exactly the same method as is used here for $\zeta(2)$.

The method involves summing a telescoping series and estimating the value of an integral asymptotically. I don't suppose the use of these techniques together to evaluate $\zeta(2)$ is new. In fact, I expect that the origin of the STEP question lies in this result, but unfortunately I cannot remember who set the question or, if it was me, where I got it from. I have not found exactly this method described in any of the usual places.‡ It is a concise, rigorous but elementary, evaluation of $\zeta(2)$ so it seemed worth an airing.

Some 14 methods of evaluating $\zeta(2)$ are given on Robin Chapman's website [3], the method most closely related to the one described here being number 12, which starts with an identity of the Fejér kernel. If Chapman's website were updated, it would no doubt include the rather elegant method of Gleb Glebov [4] and some interesting additions to the double integral method [5, 6]. It would also include the very neat proof, relying on a mean value theorem, given by Moreno [7]. I thank the referee for drawing my attention to Moreno's paper, not least because it has no fewer than 85 useful references.

No review would be complete without Timothy Marshall's proof [8], which must surely be the shortest possible. You just substitute $z = e^{i\pi}$ into the identity

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\ln z + 2\pi i n)^2} = \frac{z}{(z-1)^2}.$$

---

* Sixth Term Examination Papers are the examinations, now only in mathematics, used by some UK universities as a basis for conditional offers for students applying to study mathematics or, occasionally, other mathematical subjects; see http://www.stepmathematics.org.uk

† A surprise to me; some readers will have come across it before. See Appendix C.

‡ But see Appendix C.

The proof is essentially two lines, since the above identity can be obtained essentially by pure thought — though that requires a fairly solid understanding of complex analysis.

For the evaluation of the integral (2) which, as I mentioned above, arises as a bonus to the evaluation of $\zeta(2)$, there are of course very many methods. Nick Lord [9] recently did a great service in drawing attention to two lovely articles by G. H. Hardy [10, 11] in this journal in which Hardy scores the various methods known to him according to the level of analytical intricacy required to justify them. Some of these have echoes of the method described here, but the closest seems to be Dr Dixon's method to which Hardy awards the highest mark for analytical intricacy, an honour he surely would not have bestowed on the rather simple justifications required for the evaluation given in this article.

The STEP question

This is the STEP question, which appears as question 4 on STEP Paper II, 1998 (with x replaced by $2x$ to improve the typography, and a slight infelicity* in the definition of I_n removed):

> The integral I_n is defined by
>
> $$I_n = \int_0^{\frac{1}{2}\pi} (\pi - 4x) \sin\big((2n + 1)x\big) \operatorname{cosec} x \, dx, \tag{3}$$
>
> where n is a non-negative integer. Evaluate $I_n - I_{n-1}$ for $n \geqslant 1$, and hence evaluate I_n leaving your answer in the form of a sum.

Remembering that

$$\sin A - \sin B = 2 \cos\big(\tfrac{1}{2}(A + B)\big) \sin\big(\tfrac{1}{2}(A - B)\big) \tag{4}$$

we find that

$$I_n - I_{n-1} = \int_0^{\frac{1}{2}\pi} 2(\pi - 4x) \cos 2nx \, dx.$$

Integration by parts and a further integration gives

$$I_n - I_{n-1} = \begin{cases} \dfrac{4}{n^2} & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases} \tag{5}$$

Thus, by telescoping,

$$I_n = I_0 + \sum_{k \text{ odd}}^{n} \frac{4}{k^2},$$

* Spotted by the referee.

where the sum is over all odd positive integers less than or equal to n. However

$$I_0 = \int_0^{\frac{1}{2}\pi} (\pi - 4x)\,dx = 0,$$

so

$$I_{2n+1} = \sum_{k=0}^{n} \frac{4}{(2k+1)^2} \tag{6}$$

and $I_{2n} = I_{2n-1}$ by (5). The solution to the STEP question is now complete: perhaps not quite worth full marks under examination conditions, in view of the lack of detail in the integration by parts.

The connection with $\zeta(2)$ arises from the following elementary but rather satisfying result*. Taking the odd and even terms of (1) separately, and starting from (6), we have

$$\tfrac{1}{4}\lim_{n\to\infty} I_n = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \zeta(2) - \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \zeta(2) - \tfrac{1}{4}\zeta(2) = \tfrac{3}{4}\zeta(2),$$

which leads to

$$3\zeta(2) = \lim_{n\to\infty} I_n.$$

In order to obtain the famous $\zeta(2) = \dfrac{\pi^2}{6}$ it remains only to show that $\lim_{n\to\infty} I_n = \dfrac{\pi^2}{2}$.

This limit must exist (since we know that the sum for $\zeta(2)$ converges) despite the fact that the limit as $n \to \infty$ of the *integrand* of I_n certainly doesn't exist: the integrand just oscillates faster and faster. However, these rapid oscillations are the key: the more oscillations that are packed into the fixed interval of integration, the more nearly exact is the cancellation of the positive and negative parts of the integrand. The result of the rapid oscillation is that the contribution to the integral from most of the interval is negligible. As is so often the case in asymptotic analysis, the property of the integrand that creates problems for approximation of the integral (by the trapezium rule, for example) is exactly the property that makes the asymptotic estimation of the integral tractable.

An estimation lemma

In order to evaluate $\lim_{n\to\infty} I_n$ we use the following standard lemma from asymptotic analysis, which is, in fact, an easy case (because of the restrictions on $g(x)$) of the Riemann-Lebesgue lemma [12].

---

* A similar result gives a quick and easy proof of the divergence of the harmonic series (i.e. the non-existence of $\zeta(1)$); see Appendix B to this article.

Lemma: Let

$$J_m = \int_a^b g(x) \sin mx \, dx \tag{7}$$

where $g(x)$ is any function with continuous first derivative and a and b are finite. Then $J_m \to 0$ as $m \to \infty$.

Proof: This hardly needs proving: a sketch of the rapidly oscillating sine with relatively slowly varying amplitude $g(x)$ is convincing. However, integration by parts also does the trick rather uninterestingly, and this formal proof of our lemma is relegated to Appendix A to this article.

Evaluation of $\lim_{n \to \infty} I_n$

Even without taking the limit $n \to \infty$, the integral definition (3) for I_n looks a bit delicate: for any given n, the cosec in the integrand diverges as the lower endpoint of the integral is approached. However, for x close enough to 0, the implied ratio of sines in the integrand is, to lowest order, just the ratio of their arguments, which is finite at $x \to 0$. This argument is not rigorous because rigorous argument is not required: it is clear that nothing can go wrong at the endpoints of the integral. However, if we were keen on this sort of thing, we could define a function h by

$$h(x) = \begin{cases} \sin((2n + 1)x) \operatorname{cosec} x & \text{for } 0 < x \leqslant \frac{1}{2}\pi \\ 2n + 1 & \text{for } x = 0 \end{cases}$$

and show that it is continuous. We could then use standard results to infer that $(\pi - 4x)h(x)$ is integrable and not, after all, particularly delicate as an integrand.

In order to use our lemma, we write I_n in the form

$$I_n = \int_0^{\frac{1}{2}\pi} f(x) \sin mx \, dx, \tag{8}$$

where $m = 2n + 1$ and $f(x) = (\pi - 4x) \operatorname{cosec} x$.

Our lemma cannot be applied directly to I_n because $f(x)$ in equation (8) is not even bounded at $x = 0$, let alone continuously differentiable. However, if we changed the lower limit in the integral I_n from 0 to a, where $0 < a < \frac{1}{2}\pi$, our lemma would apply, and the new integral would tend to 0 in the limit $m \to \infty$. This shows that the dominant contribution to the integral in this limit comes from the neighbourhood of $x = 0$. For small x, we have

$$f(x) = \frac{\pi - 4x}{x - \frac{1}{6}x^3 + \dots} = \frac{\pi - 4x}{x}\left(1 + \tfrac{1}{6}x^2 + \dots\right)$$

where the bracketed expansion converges for $|x| < \pi$ (i.e. up to the next zero of $\sin x$). Accordingly, we subtract the bad behaviour at the origin by writing (trivially)

$$I_n = \int_0^{\frac{1}{2}\pi} \left(f(x) - \frac{\pi}{x} \right) \sin mx \, dx + \int_0^{\frac{1}{2}\pi} \frac{\pi}{x} \sin mx \, dx. \tag{9}$$

The idea is to use our lemma on the first integral and evaluate exactly the second, simpler, integral in the required limit $m \to \infty$.

The first of the integrals in equation (9) satisfies the conditions of our lemma, so it tends to 0 as $m \to \infty$ and hence as $n \to \infty$.

Although the integrand of the second of the two integrals in equation (9) is well behaved at $x = 0$, we again cannot apply our lemma, because $\frac{\pi}{x}$ is bad at $x = 0$. However, we can evaluate the integral exactly in the limit $m \to \infty$. Setting $u = mx$ gives

$$\int_0^{\frac{1}{2}m\pi} \frac{\pi \sin u}{u} \, du.$$

Now we take the limit $m \to \infty$ and use the familiar result

$$\int_0^{\infty} \frac{\sin u}{u} \, du = \frac{\pi}{2}, \tag{10}$$

and we are done: $3\zeta(2) = \lim_{n \to \infty} I_n = \frac{1}{2}\pi^2$.

The integral in equation (10), which looks so innocuous, is surprisingly tricky to evaluate by elementary means. It is not even obvious that it converges. Methods of evaluating it, some requiring apparently more complicated integrals such as

$$I(\alpha) = \int_0^{\infty} \frac{\sin u \, e^{-\alpha u}}{u} \, du,$$

have often been discussed in *The Mathematical Gazette*. In many cases, it is necessary to exchange the order of limiting processes. For example,

$$\int_0^{\infty} \frac{\sin u}{u} \, du = \int_0^{\infty} \lim_{\alpha \to 0} \frac{\sin u \, e^{-\alpha u}}{u} \, du \overset{??}{=} \lim_{\alpha \to 0} \int_0^{\infty} \frac{\sin u \, e^{-\alpha u}}{u} \, du.$$

It is for such exchanges, some requiring quite difficult analysis, that Hardy awards his scores in the papers [10, 11] referred to in the introduction. With our rather simple lemma, we can avoid this sort of tricky analysis.

Let

$$K_n = \int_0^{\frac{1}{2}\pi} \sin\big((2n + 1)x\big) \operatorname{cosec} x \, dx. \tag{11}$$

Using the identity (4), we find that

$$K_n = K_{n-1} = \dots = K_0 = \frac{\pi}{2}. \tag{12}$$

Again setting $m = 2n + 1$, we obtain (compare (9))

$$K_n = \int_0^{\frac{1}{2}\pi} \left(\operatorname{cosec} x - \frac{1}{x} \right) \sin mx \, dx + \int_0^{\frac{1}{2}\pi} \frac{1}{x} \sin mx \, dx.$$

Using our lemma, we see that the first of these integrals tends to 0 as

$m \to \infty$, as before. Then

$$\frac{\pi}{2} = K_n = \lim_{n\to\infty} K_n = \lim_{n\to\infty}\int_0^{\frac{1}{2}\pi}\frac{\sin mx}{x}\,dx = \lim_{n\to\infty}\int_0^{\frac{1}{2}m\pi}\frac{\sin u}{u}\,du = \int_0^{\infty}\frac{\sin u}{u}\,du,$$

which is what we want. Note: no exchange of order of limiting processes was used!

Appendix A: Proof of our lemma

Referring to equation (7), we have

$$J_m = -\frac{1}{m}\left(g(b)\cos mb - g(a)\cos ma\right) + \frac{1}{m}\int_a^b g'(x)\cos mx\,dx. \quad (13)$$

The first of these two terms clearly tends to 0 as $m \to \infty$, so we only need to bound the integral and we are done. By a standard result for integrals, we know that

$$\left|\int_a^b g'(x)\cos mx\,dx\right| \leqslant \int_a^b \left|g'(x)\cos mx\right|dx.$$

Furthermore,

$$\int_a^b \left|g'(x)\cos mx\right|dx \leqslant \int_a^b \left|g'(x)\right|\left|\cos mx\right|dx \leqslant \int_a^b \left|g'(x)\right|\,dx \leqslant (b-a)\,G,$$

where G is the largest value of $|g'(x)|$ on the interval $[a, b]$. This is certainly bounded as a function of m since it does not depend on m.

Thus $J_m \to 0$ as $m \to \infty$.

Appendix B: Proof that the harmonic series diverges

This proof by contradiction is not new, but it is probably not widely known.*

Assuming that the series converges, let

$$\zeta(1) = \sum_{n=1}^{\infty}\frac{1}{n}.$$

Decomposing into odd and even terms, we have

$$\zeta(1) = \sum_{k=1}^{\infty}\frac{1}{2k-1} + \sum_{k=1}^{\infty}\frac{1}{2k} > \sum_{k=1}^{\infty}\frac{1}{2k} + \sum_{k=1}^{\infty}\frac{1}{2k} > \zeta(1),$$

which gives the contradiction. The only possible conclusion is that the convergence assumption was incorrect.

* The referee has supplied an enjoyable reference [13] which gives a list of proofs of the divergence of the harmonic series; the proof given here is number 8 out of 20.

Appendix C: Some contributions from the referee

In addition to some helpful corrections and suggestions, including an application of the method presented here to the integral

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2 nx}{\sin^2 x}\, dx$$

leading to the result $\int_0^{\frac{1}{2}\pi} \frac{\sin^2 t}{t^2}\, dt = \frac{1}{2}\pi$, the referee made the following observation. If we split up my integral I_n, giving

$$I_n = \pi K_n - \int_0^{\frac{1}{2}\pi} 4x \sin\left((2n + 1)x\right) \operatorname{cosec} x \, dx, \tag{14}$$

where K_n is defined in equation (11), we can use the evaluation of K_n given in (12) to get the required $\frac{1}{2}\pi^2$, then use our lemma immediately to dispose of the integral in (14) in the limit $n \to \infty$. The referee then notes that this is essentially the proof given by Daniel Giesy in reference [14], of which I was unaware. Had I unearthed Giesy's proof in my, as I thought, rather thorough search of the literature I would perhaps not have embarked on my own contribution to this well-ploughed but fertile field. However, I do think the asymptotic approach is just a bit different from what already exists in the literature and is therefore worth recording.

The referee also observes that the Riemann-Lebesgue lemma, a simple case of which is the basis of this approach, is also the (not usually explicitly stated) basis of the many evaluations of $\zeta(2)$ which use Fourier series. This is of course true: the Fourier inversion theorem and Parseval's theorem make use of the Riemann-Lebesgue lemma.

Finally, the referee drew my attention to papers [15] and [16] in *The Mathematical Gazette*, which may have been the inspiration for the original STEP question.

References

1. See https://maths.org
2. http://stepdatabase.maths.org
3. Robin Chapman, Evaluating $\zeta(2)$, http://empslocal.ex.ac.uk/people/staff/rjchapma/etc/zeta2.pdf (2003)
4. Gleb Glebov, A peculiar proof of an identity of Euler, *Math. Gaz.* **99** (March 2015) pp. 139-144.
5. G. J. O. Jameson and Nick Lord, Evaluation of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by a double integral, *Math. Gaz.* **97** (November 2013) pp. 503-505.
6. Tim Jameson, Another proof that $\zeta(2) = \frac{\pi^2}{6}$ by double integration, *Math. Gaz.* **97** (November 2013) pp. 506-507.

7. Samuel G. Moreno, A one-sentence and truly elementary proof of the Basel problem, *The College Mathematics Journal* **47** (2) (March 2016) pp. 134-135.

8. Timothy Marshall, A short proof of $\zeta(2) = \frac{\pi^2}{6}$, *American Mathematical Monthly* **117** (4) (April 2010) pp. 352-353.

9. Nick Lord, Euler, the clothoid and $\int_0^\infty \frac{\sin x}{x}\,dx$, *Math. Gaz.* **100** (July 2016) pp. 266-274.

10. G. H. Hardy, The integral $\int_0^\infty \frac{\sin x}{x}\,dx$, *Math. Gaz.* **5** (June-July 1909) pp. 98-103.

11. G. H. Hardy, Further remarks on the integral $\int_0^\infty \frac{\sin x}{x}\,dx$, *Math. Gaz.* **8** (July 1916) pp. 301-303.

12. http://mathworld.wolfram.com/Riemann-LebesgueLemma.html

13. S. J. Kifowit and T. A. Stamps, The harmonic series diverges again and again, http://stevekifowit.com/pubs/harmapa.pdf

14. D. P. Giesy, Still another elementary proof that $\sum \frac{1}{k^2} = \frac{\pi^2}{6}$, *Math. Mag.* **45** (1972) pp. 148-149.

15. F. Chorlton, Summation and properties of $\sum \frac{\cos nx}{n^k}$ and $\sum \frac{\sin nx}{n^k}$, *Math. Gaz.* **79** (July 1995) pp. 368-371.

16. F. Chorlton, Evaluation of integrals by difference methods, *Math. Gaz*. **67** (June 1983) pp. 133-136.

10.1017/mag.2018.17

STEPHEN SIKLOS
Jesus College, Cambridge CB5 8BL
e-mail: *stcs@cam.ac.uk*