102.11 Four congruent right triangles

The following (somewhat surprising) result about a square is possibly not as well-known as it should be.

Result (Square): The diagram below shows four directly congruent right-angled triangles *PAB*, *QCB*, *RCD* and *SAD*, with right angles at *P*, *Q*, *R* and *S*, such that *ABCD* is a square.

In this configuration *P*, *Q*, *R* and *S* are collinear.

(Two triangles are *directly congruent* if neither of them needs to be turned over to establish the congruence. Points are *collinear* if they lie in a straight line.)

The reader may like to try to prove this result before reading on.

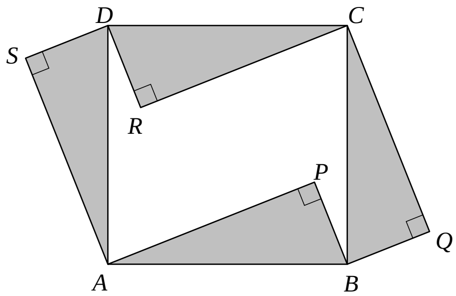

FIGURE 1

Is the result still true when the square is replaced by a rhombus? In that case, the triangles may overlap, as shown in Figure 2.

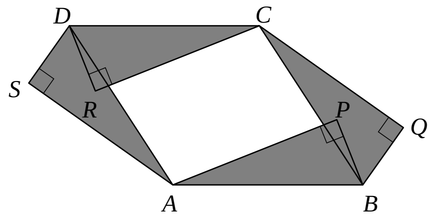

FIGURE 2

We give an elementary proof of this generalisation, in which *ABCD* is a rhombus and the triangles may overlap. Thereby we automatically prove the result for a square.

Result (Rhombus):

Figure 3 shows four directly congruent right-angled triangles *PAB*, *QCB*, *RCD* and *SAD*, with right angles at *P*, *Q*, *R* and *S*.

In this configuration *P*, *Q*, *R* and *S* are collinear.

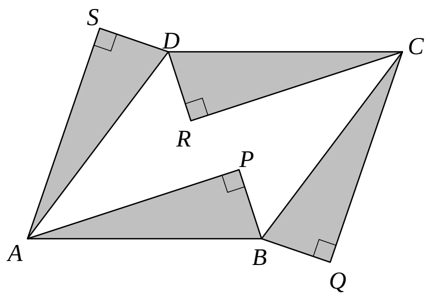

FIGURE 3

Proof: From the congruency, $PB = BQ$ and $\angle CBQ = \angle ABP$, so that $\angle PBQ = \angle ABC$. Join P to Q, as shown in Figure 4, and consider the marked angle of the isosceles triangle PBQ.

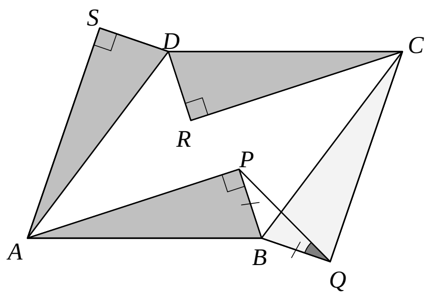

FIGURE 4

We have

$$\angle BQP = \tfrac{1}{2}\left(180^\circ - \angle PBQ\right)$$
$$= \tfrac{1}{2}\left(180^\circ - \angle ABC\right).$$

In a similar way,

$$\angle RQC = \tfrac{1}{2}\left(180^\circ - \angle BCD\right).$$

Therefore

$$\angle BQP + \angle RQC = 180^\circ - \tfrac{1}{2}\left(\angle ABC + \angle BCD\right).$$

But AB and DC are parallel, so that

$$\angle ABC + \angle BCD = 180^\circ.$$

Hence

$$\angle BQP + \angle RQC = 90^\circ$$
$$= \angle BQC.$$

It follows that P, Q and R are collinear.

By repeating this argument 'on the other side', it follows that P, R and S are collinear. Therefore P, Q, R and S are collinear, as claimed.

Note that exactly the same argument applies whether or not PQ is shorter than QR, as in all our diagrams, or if the triangles overlap.

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