Nurturing the next generation of mathematicians: the case for mathematical fluency

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Twelve months ago, when I was listening to Mike Askew's Presidential Address, my emotions were divided between thoroughly enjoying his presentation and a growing sense of trepidation that it was my turn next! The time seems to have passed very quickly and I'm somewhat nervous, but incredibly honoured, to take the helm as we explore "the case for mathematical fluency".

Let's begin by doing some mathematics together. I've left a couple of packs of objects on each table*. Take them out and see what you notice about them. What do you think the question might be?

Working with other members on your table, the challenge is for each group to accurately mark the centre of each circular object.

If we rewind for a moment, I first came across this challenge as part of a cross-curricular topic with a group of 7 to 11-year-olds who had been tasked

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* Each pack contains metal coins, paper side plate, cardboard drinks mat, crockery dinner plate, plastic lid, paper doily, wooden wheel for model car and so on.

with constructing a moving vehicle from materials close to hand – cardboard, straws, glue, wood offcuts and some pre-cut wooden wheels. As the bodies of their vehicles slowly took shape, their respect for the mantra ‘measure twice, cut once’ rapidly increased as stocks of wood ran perilously low. Anyway, once their measuring skills had sufficiently improved to ensure completion of the bodywork without resorting to a new stock order, attention turned to constructing the chassis. At this point the class discovered that the wooden wheels which had been supplied were solid, the students would need to drill a hole in the centre of each wheel to fix their axles. Eager to get going, they quickly discovered that guessing where the centre of the wheel might be located led to off-centre holes and an early-than-planned introduction to the working of cams. Problem-solving mode was needed and their ideas came thick and fast; swapping to paper wheels would have enabled them to fold the wheels twice to establish the central point, but paper wheels were deemed insufficient to take the load so that idea was, somewhat reluctantly, shelved. At this point, the students were stuck despite their willingness to ‘try, try and try again’; they had tried drilling holes, but without an accurate centre point the ride was not smooth and they had considered changing material to paper but knew that their models would not take the load. Getting stuck is a key experience for any mathematician, and we know that ensuring our students develop strategies to manage such situations is a crucial part of our work. We all solve problems in different ways and at different rates. We get stuck. Whether it was this problem or another recent challenge, how do you respond to getting stuck? Have you always felt that way, even when you were at school? How might your classes respond to being stuck? Why might they respond that way?

Being a mathematician is surely about so much more than recalling facts? Our subject has the potential to bring moments of ‘awe and wonder’ into our classrooms and lecture theatres. This brings us to our main focus today: fluency. Fluency was the theme of my doctoral thesis so I've spent a great deal of time thinking about it but what does fluency mean to you? During my research, I've encountered various types of fluency. What's the same and different about them? I suggest that arithmetic/computational fluency would be limited to number work whereas algorithmic/procedural fluency seem to go further by encompassing the wider mathematical curriculum. For example, correctly applying a procedure for bisecting an angle might indicate algorithmic fluency in that aspect of geometry. However, comparing both algorithmic fluency and computational fluency with mathematical fluency seems trickier to me. I suggest that mathematical fluency arguably requires proficiency in several interconnected and interdependent ‘strands’, which was captured in the ‘rope model’ of Figure 1 in [1, pp. 115-135].

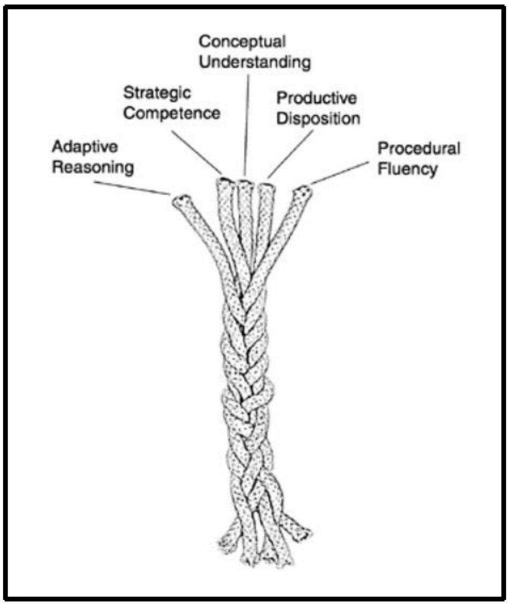

FIGURE 1: The 'rope model' illustrating the five strands of mathematical proficiency [1]

I strongly believe that the rope model is a powerful visual for illustrating the interconnected aspects of mathematical fluency; it recognises the importance of developing *procedural fluency* alongside *conceptual understanding* among learners of mathematics, but it suggests that there is so much more to consider when nurturing young mathematicians for them to become fluent.

Procedural fluency –	the skill in carrying out procedures flexibly, accurately, efficiently and appropriately
Conceptual understanding –	the comprehension of mathematical and scientific concepts, operations, processes and relations
Strategic competence –	the ability to formulate, represent, and solve mathematical problems, and to plan and implement inquiry into scientific questions
Adaptive reasoning −	the capacity for logical thought, reflection, explanation, and justification
Productive disposition −	the habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy

There is the need to nurture *strategic competence* so, fluency goes beyond problem-solving (even though Ofsted says it does not see sufficient non-routine problem-solving in our schools) but it is also about problem-solving and being creative.

Also, we need to nurture *adaptive reasoning*. Crucially, the rope model calls for educators to instil a *productive disposition* towards mathematics.

Isn't this something we want to instil in all our learners? An understanding about why mathematics is so important for us all, and why some of us choose to study it? Otherwise, studying mathematics can seem pointless to many learners who tire of learning facts without the opportunity to apply them, whether it's finding the centre of wheel or working out the popular pocket money challenge that we recently featured on NRICH.

> Which of these alternatives would you prefer?
>
> £10 every day
>
> £3 on the first day, £3.50 on the second, £4 on the third, and so on, increasing by 50p per day
>
> choosing 1p on the first day, 2p on the second, 4p on the third, and so on, doubling each day

You can enjoy reading some students' thoughts on these options see [2].

Often classroom mathematics can resemble endless drills without sharing the moments of ‘awe and wonder’ which make our subject so special. If we compare such an approach towards teaching and learning mathematics with football, it would be like restricting young players to just practise dribbling through cones, without sharing the wonder of yet another Messi goal. Or, in music, only permitting the rehearsal of scales without ever listening to some of our greatest ever compositions. Mathematics is an active subject, and being fluent requires so much more than understanding and following algorithms; it needs problem-posing, decision-making and an appreciation of the power of mathematics.

We are told that students should “become fluent in the fundamentals of mathematics, including through varied and frequent practice with increasingly complex problems over time, so that pupils develop conceptual understanding and the ability to recall and apply knowledge rapidly and accurately” [3]. If we look more carefully at that statement, accuracy and efficiency seem highly valued. Yet we know that working too quickly can lead to errors and it can cause stress in some children. Who decides if an approach is efficient? I suggest that mathematical fluency requires decision-making too – by making decisions you have the opportunity to be creative, set off on a mathematical adventure and derive a sense of enjoyment too. Perhaps even more importantly, choosing the most suitable approach can reduce avoidable errors too!

Enabling opportunities for students to take decisions, by having both the knowledge and the willingness to work flexibly rather than blindly follow an algorithm, are key ways forward. Have a look at this NRICH activity Fruity Totals (see Figure 2) [4].

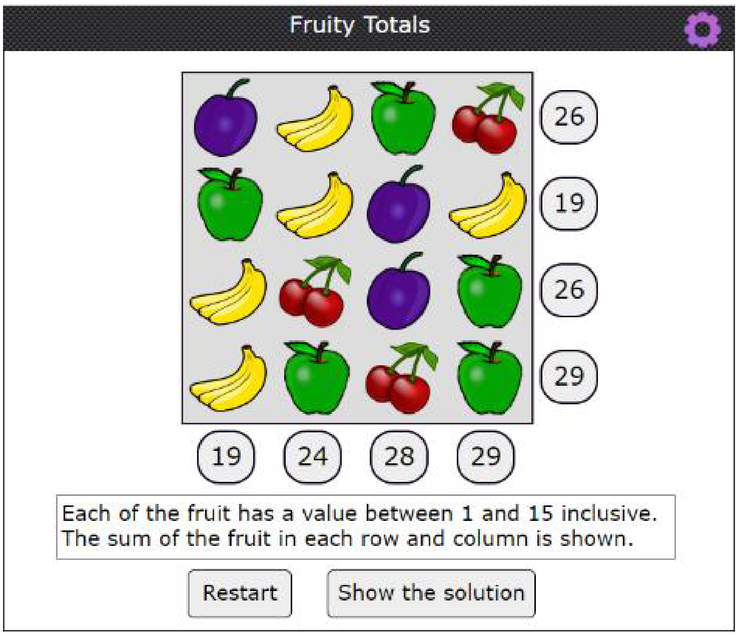

FIGURE 2: Fruity Totals

Think back to your initial approach to this problem. How did you get started? Did everyone start in the same way? Did anyone get stuck? Getting stuck is an important aspect of doing mathematics but we're often told to 'try, try and try again'. Sometimes, trying the same thing again does not help. Being flexible does, it gives students another way forward, it also allows them to check their workings. Becoming a resilient learner requires students to experience activities where they can get stuck, then draw on their willingness to try another approach, and sometimes another approach after that too!

Let's look at an example that I explored with some primary-aged students. I asked them to subtract 695 from 702. Take a look at Kate's working out:

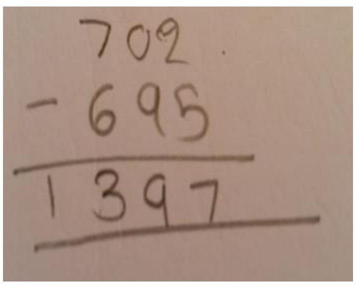

FIGURE 3: Kate's working out for the calculation 702 – 695

What do you notice about her answer? (Added rather than subtracted). Kate did not realise her error until we met the following day after she'd completed the workbook. She'd done each of the four subtractions in a similar way, adding rather than subtracting. She had not checked her work, but soon realised her error when we went through the calculations together. How might you expect an 8-year-old to do this calculation? How would you like them to do it? How about an 11-year-old? And, a 14-year-old? What about an adult? At primary school, we often reward our students for learning their formal calculation strategies (which research highlights that are highly prized by society e.g. [5] Cockcroft 1982), but we do not always reward learners for their decision-making skills which I suggest are even more important. For me, it's like having a toolbox packed with different screwdrivers – each has a specific use and it's knowing when to use them, as well as when NOT to use them, that's important. It's the same with calculation strategies. A Year 6 student Kevin could quickly work out the answer to $702 - 695$ in his head but he chose the more complicated formal algorithm in class, "You have to write it down to get a good mark." Is this fluency? Encouraging children to explain their thinking, rather than simply follow an algorithm, is surely a very useful foundation for their later studies when problem-solving at GCSE and beyond? However, SATS hold an enormous sway over decision-making at primary school.

Mark schemes currently reward using the formal algorithm rather than decision-making skills, but it does not need to be like that. We could explore other ways of working and assessing. What about estimating? It's a key skill for using mathematics in real life situations but it is rarely tested, and therefore usually under-valued, in our schools. When should students use a calculator? Could they check their answers using a calculator to help improve their improve fluency? There are other approaches which might

Award **TWO** marks for the correct answer of 215,016

If the answer is incorrect, award **ONE** mark for the formal method of long multiplication with no more than **ONE** arithmetic error, e.g.

- $$\begin{array}{rr} & 3468 \\ \times & 62 \\ \hline & 6936 \\ & 208080 \\ \hline & 214016 \end{array} \textit{ (error)}$$

OR

- $$\begin{array}{rr} & 3468 \\ \times & 62 \\ \hline & 6934 \; \textit{(error)} \\ & 208080 \\ \hline & 215014 \end{array}$$

FIGURE 4: Mark scheme for 2019 Y6 SAT calculation 3468×62 [6]

help students to become much more fluent. When problem-solving, teachers are often faced by a wall of hands and the plea “Just tell us what to do!” Could we adopt a different approach? In my research I came across the Chelsea diagnostic approach (work done by former MA President Margaret Brown and her team) of asking learners to identify the correct calculation for a given problem.

That said, we also need to recognize that some students (and adults) enjoy following algorithms. Train spotters have feelings too, as Paul Andrews argued in his (ATM!) response to Dave Hewitt's *Trainspotter's Paradise*. In my own research one 11-year-old student talked about his favourite algorithm, I wonder if you can work out which it is: “It's my favourite method to use, like no other methods so far have even come close for me. I enjoy doing it more because it's simple and if you're not sure on an answer you can always do it again. You can look back, see what you did wrong. When you do that, it's not too hard to get hold of and understand.” Edward, who was doing the talking, was aged 10 when we met. Which algorithm do you think he was describing as his favourite? (Long division.)

Although mark schemes might not explicity promote estimating or checking answers, it was clear from my research that it was happening in some classes. In most classes, though, the learners suggested that they checked their work by scanning the numbers in their working out, not by applying another strategy or comparing their answer to an estimate. Knowing more strategies enables flexibility. Too often children repeated the same errors because they did not have efficient ways to check their work. Polya, that great Hungarian mathematician, talked about the importance of reflecting on our answers, but to what extent do we value reflection in our classrooms and assessment systems?

There were some green shoots in my research. Alex was asked to calculate 693 divided by 3. What did he do here?

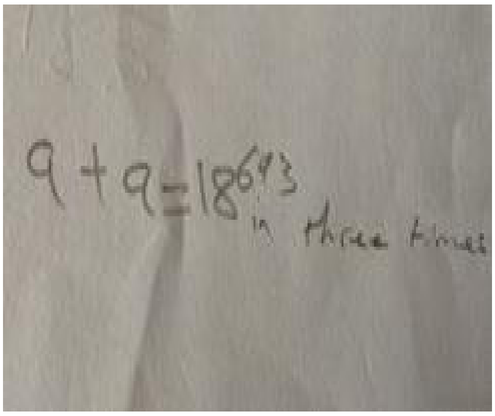

FIGURE 5: Alex's annotations which appeared beside his calculation 693 ÷ 3

Alex explained that he used his knowledge about divisibility rules to check his answer; he totalled the digits in the dividend 693, getting 18, which he knew was divisible by 3. Divisibility rules are a useful, efficient tool for checking answers. Inverses can be useful too, but they were rarely seen in the work of the children I studied. Here's a rare example from my study (Figure 6). Can you see how the student checked their work for calculating 517 divided by 19?

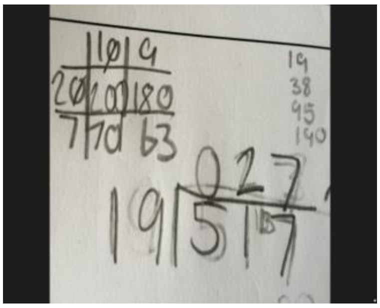

FIGURE 6: Checking a calculation using the inverse operation

As you may have already noticed, the student checked their division calculation using the inverse operation of division by multiplying their divisor and dividend together.

However, although many primary-aged children are still learning their calculation strategies, this room is packed with MA numbers who may already know many ways to complete a calculation (some more efficient and less likely to lead to errors than others). So, here's a challenge for you.

How many ways do you know to calculate 4987 divided by 35?

- How well can you explain how each method works?
- What's the same and different about them?
- Which is the most efficient method for you? Is that the same for everyone on your table?

Now let's look at an example of a different strategy which I do not believe has been mentioned so far today (but apologies if you have already been using it today). There's quite a lot of writing, can you figure out the calculation they were trying to work out?

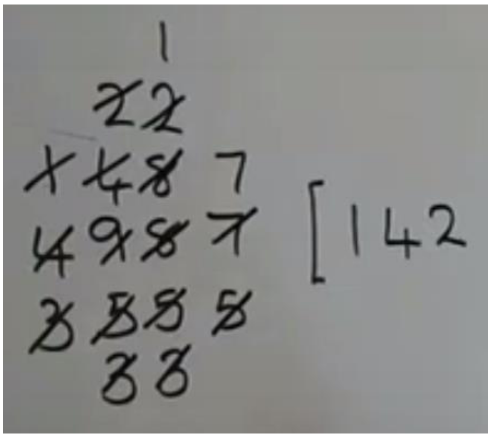

FIGURE 7: Taken from Galley Division [7]

(Same calculation as before i.e. 4987 divided by 35.) Sometimes there are calls for students to focus on a single strategy for each operation, rather than developing flexibility. In debates regarding the relative merits of the grid method and long multiplication, or chunking and long division, it seems easy to assume that both long multiplication and long division have always been the ‘preferred’ options for those operations. Not so! In the 1600s, students may have been learning the Galley method for division (Figure 7). It's well worth spending some time exploring this approach before we go any further.

I understand that Galley division is still taught in parts of North Africa, which is where I was born, and Middle East. I first came across it when I was doing some research in my former role as a mathematics consultant in Lincolnshire. I came across the video I just shared with you on a website called NRICH, and we all know where that led…ten years later I was appointed as NRICH's Director. All I can say, be careful what you click on! More seriously, having the curiosity to explore different ways of performing calculations and developing a knowledge of multiple strategies, and a willingness to choose between them for individual circumstances, is a key step on the road to becoming a fluent mathematician. As we saw with the calculation $702 - 695$, relying too heavily on a single approach can result in inefficient calculations as well as leading to avoidable errors. Moreover, having alternative strategies is a very useful way for getting ‘unstuck’. At primary level, we can encourage our students to become more fluent by ensuring they take each calculation or problem on its individual merits, rather than adopting a ‘one size fits all’ approach. Developing a willingness to work flexibly, and share their reasoning, will offer excellent preparation for further problem-solving work at secondary level as well as enabling them to make connections across different areas of the curriculum:

> The reason that one problem can be solved in multiple ways is that mathematics does not consist of isolated rules but connected ideas. Being able to and tending to solve a problem in more than one way, therefore, reveals the ability and the predilection to make connections between and among mathematical areas and topics. (Ma, 1999) [8]

Flexibility is essential for mathematics. A willingness to work flexibly enabled Euler to solve the Königsberg bridge problem, where citizens tried in vain to find a route along bridges between islands that would bring them back to the beginning. A willingness to work flexibly also enabled Florence Nightingale to realise the potential of visuals for more effectively communicating vital data relating to the loss of troops during the Crimean War.

Mathematical fluency does not simply require the knowledge of algorithms and an understanding of the concepts behind them. It goes much, much further than that. It also requires our students to make decisions so that they work efficiently by choosing the most appropriate method each time, rather than adopting a ‘one size fits all’ approach. Fluency comes in many shapes and forms: Messi scores a wonder goal; Charlotte Church sings like an angel. In mathematics, fluency occurs when students embrace the power of mathematics to ask questions, explore different approaches and begin to appreciate the potential of our subject to truly produce moments of ‘awe and wonder’.

Thank you.

References

1. J. Kilpatrick, J. Swafford, B. Findell, *Adding it up: helping children learn mathematics*, The National Academies Press, Washington DC (2001).
2. Pocket Money from nrich.maths.org/13687
3. Department for Education (DfE). (2013). National curriculum in England: Framework document. Retrieved from https://www.gov.uk/government/publications/national-curriculum-in-england-primary-curriculum
4. Fruity Totals from nrich.maths.org/fruity
5. W. H. Cockcroft, *Mathematics Counts*, London: HM Stationery Office (1982).
6. Department for Education. (2020). 2019 National Curriculum Tests − Key Stage Two − Mathematics Test Mark Schemes. Accessed from https://assets.publishing.service.gov.uk/government/uploads/system/uploads/attachment_data/file/804063/STA198219e_2019_ks2_mathematics_Mark_schemes.pdf
7. Galley Division from nrich.maths.org/6276
8. L. Ma, *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*, Mahwah, NJ: Lawrence Erlbaum Associates, Inc. (1999).

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