Questions Pupils Ask!

Is calculus exact?

by **Colin Foster**

Some years ago, I read an account by the broadcaster Edward Stourton (2008) of how he became disenchanted with mathematics at school:

"I abandoned mathematics when I was fifteen because I was asked to work out the gradient of a curve. This, of course, is impossible to do with total precision, because it is in the nature of a curve to change its gradient constantly. So we were taught a technique from what is known as calculus; it involved working out an approximation so near the non-existent truth that it could just as well serve as the answer to the question. But that was not good enough for me – I was full of that unforgiving certainty of the adolescent … The whole point of mathematics, I felt, was that it delivered certainty; ambiguity I could find aplenty in literature and the arts. I chucked in the idea of a maths A level and focused on Latin; at least there you know what the rules were (pp. 69–70)."

He goes on to say that he now feels that he was wrong about this – but wrong because calculus has important real-world applications. He does not say that he was wrong about calculus being an approximation; indeed, he says that he has "learnt to accept that you cannot always have absolute answers in science and mathematics" (p. 70).

I find this disturbing, because to me calculus *is* exact, and it seems really sad that Stourton came away from his school mathematics lessons with the impression that it is only an approximation – albeit a good enough one for all practical purposes. After reading this, I found myself wondering whether school students I have taught calculus to might have been left with the same misconception – and I think it is very likely that they could have.

It is common practice to introduce the ideas of differentiation in visual ways using technology, by calculating the gradients of chords between a point of interest and a succession of ever-closer neighbours (on either side), in order to show that the values appear to get closer and closer to a particular value as the chords get closer and closer to the tangent at the point (Note 1). I have no doubt that this is very helpful for building up a sense of what it means to talk about the gradient of a curve at a point, and what differentiation is all about, but none of it convinces us that we are obtaining the precise value of the gradient at that point, or even that such a thing could be possible.

It is also common to do the same thing numerically. Typically, students might begin with $y = x^2$. Then you choose the point (3, 9), say, and draw a chord to $(3.1, 3.1^2)$, and then $(3.01, 3.01^2)$, and then $(3.001, 3.001^2)$, and so on. Using $y = x^2$ has the advantage that when you calculate the gradient of each chord you don't have to fiddle around with calculators or spreadsheets, because you can use the difference of two squares to simplify the gradients. Since $\frac{x_1^2 - x_0^2}{x^1 - x^0} = x^1 + x^0$, then we can write the gradients of the chords as

$$\frac{3.1^2 - 3.2^2}{3.1 - 3} = 3.1 + 3,$$

$$\frac{3.01^2 - 3^2}{3.01 - 3} = 3.01 + 3,$$

$$\frac{3.001^2 - 3^2}{3.0001 - 3} = 3.001 + 3, \cdots$$

Not only does this save some fiddly computation, it *kind of* shows *why* the gradient at 3 must be twice 3. And, because we haven't got too bogged down in the numbers, it's clear here that there's nothing special about 3. We can treat the '3' as a symbol for any number, since we never used anything about the threeness of the 3 in what we did.

This is all very nice, and helpful, I think, but it's perfectly possible for students to go away from this thinking that the gradient is always a little bit *more* than $2x$. The gradient at $x = 3$ is basically 6, but *plus a tiny bit*. Of course, we can address this by repeating the whole thing going to the *left* of (3, 9), and so find that the gradient there is always a little bit *less* than 6. But then what? Do we now conclude that "When x is *exactly* equal to 3, the gradient is *exactly* equal to 6"? That feels like a really massive step.

Using algebra doesn't resolve this. We can work out that the gradient of the chord joining (x, x^2) to ($x + \delta x$, $(x + \delta x)^2$), where δx can be positive or negative (but not zero), will be $2x + \delta x$. And then we can talk about "tending towards zero", whatever that means, and say that as $\delta x \to 0$ the gradient $2 \to x$. But, just as before, the gradient is never *exactly* $2x$ unless we substitute $\delta x = 0$, which we know we aren't allowed to do, because then we will end up with $\frac{0}{0}$ (Note 2).

It's easy to say that this is good enough for sixth formers, and that a really rigorous justification is just not possible until first-year analysis at university. But one problem with that is that the vast majority of students don't go on to do a mathematics degree, and so never get to analysis. What can we do for them? Are they destined to be left, like Stourton, thinking that the whole thing is just a gigantic approximation?

Although I would avoid getting into epsilons and deltas and formal definitions at this level, I do try to use helpful language, like "arbitrarily close", or "as close as you like to", which I think is much better than "closer and closer to". For example, if $y = \frac{1}{x}$ gets "closer and closer" to zero as x tends to infinity, then it also gets "closer and closer to" any number *less than zero*. "Closer and closer to" doesn't capture the idea of a limit. The important point is that y gets "as close as you like" to zero; i.e. if you decide how close to zero you would like y to get, provided you make x large enough you can always get even closer than that. The epsilon–delta definition of a limit is often presented as playing a game where, "You tell me how small you want the error to be, and I'll tell you how close to the x value you have to be so that the error will always be smaller than that". So, for our $y = x^2$ example, the challenge would be "You tell me how close to 6 you want the gradient to be, and I'll tell you how close to 3 you have to make the x to achieve that". The point is that no matter how small a difference from 6 you want to have, I can always get you closer than that by setting the x value sufficiently close to 3. Isn't that good?

But I find that this argument never really works very well with students. If they are thinking about the ideas and engaging in the argument, they always say something like, "I want to be *exactly* at 3. I don't want a *small* error – I want no error at all!" If I say something like, "You can make your error as small as you like", then they say "I'll have it as zero then, please!", and then I have to say, "Well, as small as you like, *but not actually zero*," which seems to support them in the Stourton view that calculus is really very, very accurate but not totally 100% accurate. The takeaway message is that there's always a tiny little error, and you can never completely get rid of it (but you can make it small enough to be insignificant for practical purposes).

Actually, this problem is not specific to calculus; it is the same issue with any limit. You can show students by shading in fractions of a unit square that $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots$ is pretty close to 1, but for any finite number of terms there is always a tiny bit of the square left unshaded. We can make this bit of leftover square as small as we like by taking enough terms. However small you want to make it, I can tell you how many terms you need to take to make it smaller than that. But then students come away thinking that $\sum_{i=1}^{n}\frac{1}{2^i}<1$, which, of course, is true for any *finite n*. The issue is what happens as $n \to \infty$. Do we want to say that when you "go to infinity" that somehow makes everything all right and that the partial sum then *equals* 1 exactly? Why should they believe this?

Part of the problem here is in thinking of infinity as a process that happens in time – the "..." at the end is being taken to indicate that someone keeps on adding more and more terms. If you think of it like this, then you will never get all the way to 1, because the series will never be finished. But that's not what the notation means; it is referring to the *entire* infinite sum, as a complete finished object.

One way to think about the size of the gap is to say that if it is non-negative and we can make it smaller than any positive number that we care to name, then it must be zero (Note 3). We can make the gap smaller than any positive number, just by taking enough terms of the sequence. The student may ask us how many terms it would take to do this, and then what would we say – "infinity"? It's tempting to say something like "It turns out that if you take infinitely many terms, then the gap becomes *exactly* zero", with maybe a "You'll prove it at university", and that is either taken on trust or rejected as false. But thinking about the gap we can ask "Can you think of a non-negative number that is smaller than any positive number?" The answer must be zero, but of course, we had to sneak in "non-negative" there! It may seem a bit weird, because in epsilon–delta terms we are saying that we can make epsilon smaller than any positive number, so it can be zero, but we actually began by saying that epsilon had to be strictly greater than zero.

Perhaps this isn't really something that we can demonstrate at any level, even at university. We can formalize it in terms of the *completeness axiom*, which is equivalent to accepting that 0.99999... = 1. Lots of people have written about the difficulty of persuading students about this, often seeming to blame pupils for having trouble with it. Pupils accept that fractions have non-unique representations (e.g., $\frac{2}{3}=\frac{6}{9}$) but not decimals (e.g., $0.\dot{9}=1$). I have often asked pupils whether they believe that 0.33333... = $\frac{1}{3}$, and "Is that exact or just approximate?", and I find that they mostly say that it's exact. So then I multiply both sides by 3 to obtain 0.99999... = 1. Multiplication of infinite decimals can sometimes be problematic, because you want to start at the right-hand side, not the left, but here it is clear that there will be no carrying at any point – each 3 simply turns into a 9. So they get 0.99999... = 1. But I usually find that they don't believe it – they are looking for the trick. Eventually they may end up questioning 0.33333... = $\frac{1}{3}$. After all, they may say, however many 3s you write down, there is always a "remainder 1" on the end, and that's never going to disappear – so saying that "It's just a line of 3s" is an approximation, because you're ignoring that remainder 1. So the result of the discussion can be that they stop believing that 0.33333... = $\frac{1}{3}$ is exact, which, I suppose, is at least consistent! But it's the same issue again about the "gap", and I think in this case we just have to say that this is an axiom, that it makes sense to accept, rather than something that we can demonstrate is true.

References

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Notes

1. It is important to realize that as a point slides down the curve, towards the fixed point, and the chords get shorter and shorter, there is a *change of state* when the points coincide, and a *chord* becomes a *tangent*.
2. I am grateful to Bob Burn for pointing out an alternative approach: All lines (except $x = 1$) through (1, 1) have the form $y - 1 = m(x - 1)$. This generally cuts $y = x^2$ at two points, but at only one point when $m = 2$. So this is the tangent.
3. This assumes that we don't include infinitesimals! (For a historical view, see Kleiner, 2001.)

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Author Colin Foster, School of Education, University of Leicester, 21 University Road, Leicester LE1 7RF.
e-mail: **c@foster77.co.uk**
website: **www.foster77.co.uk**