The Root of the Problem: A Brief History of Equation Solving

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Background and References

“Mathematics is a unique aspect of human thought, and its history differs in essence from all other histories.”
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- *Bluff Your Way in Maths* R. Ainsley

Notation

- RHETORICAL   Rhind Papyrus c. 1650 BC

  Of four pipes, one fills the cistern in one day, the next in two days, the third in three days, the fourth in four days: if all run together, how soon will they fill the cistern?

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  Of four pipes, one fills the cistern in one day, the next in two days, the third in three days, the fourth in four days: if all run together, how soon will they fill the cistern?

Let C be the amount of water required to fill the cistern. In one day, all pipes together will produce an amount of water equal to

$$C + \frac{C}{2} + \frac{C}{3} + \frac{C}{4} = \frac{25}{12}\,C,$$

so the amount of time required to fill the cistern is $\frac{12}{25}$ of a day, that is, 11 hours 31 minutes and 12 seconds.

Notation

- SMC MATHEMATICAL CHALLENGE 2007

  Calum was floating down river on a raft, when, half a mile downstream, his brother Duncan set off in a canoe. Duncan paddled downstream as quickly as he could, then turned round and paddled back again, still at his best pace. He arrived back at his starting point just as Calum floated by. Assuming Duncan's best pace in still water is 10 times that of the river current, how far did he paddle?

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ANSWER: $\frac{99}{20}$ miles.

Notation

- SYNCOPATED Diophantus c. 250 AD

1	2	3	4	5	6	7	8	9	10
α	β	γ	δ	ϵ	ζ	ξ	η	θ	ι

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$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \alpha & \beta & \gamma & \delta & \epsilon & \zeta & \xi & \eta & \theta & \iota \end{array}$$

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unknown cubed 1, unknown squared 13, unknown 8

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$$\kappa^{\Upsilon}\alpha\Delta^{\Upsilon}\iota\gamma\varsigma\eta$$

Notation cont.

- SYMBOLIC

$+, -$	Widman	1489
$\sqrt{}$	Rudolf	1525
$=$	Recorde	1557
unknowns=vowels knowns=consonants	Viète	1570
unknowns=late letters knowns=early letters	Descartes	1630
$>, <$	Harriot	1631

Notation cont.

$\times$, $\sim$, π	Oughtred	1640
∞	Wallis	1650
$f(x)$	Euler	1750
$n!$	Kramp	1800
$x \equiv y \mod n$	Gauss	1801

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“Mathematics is a game played according to certain simple rules with meaningless marks on paper.”
Hilbert (1862-1943)

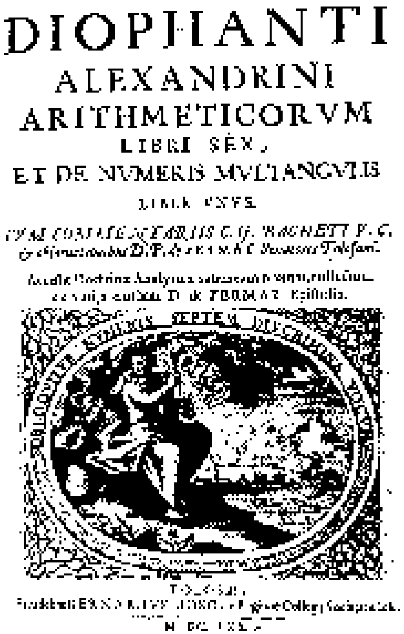
Archimedes (287 BC - 212 BC)

- Greek mathematician and astronomer
- lived and worked in Syracuse
- invented war machines
- *On Floating Bodies*
- slain by a Roman soldier

On the Measurement of the Circle

Geometric methods for calculating square roots using circles with circumscribed hexagons (similar to the Babylonians)

Diophantus (c. 200 BC - 284 BC)



- Greek mathematician
- lived and worked in Alexandria
- allowed positive rationals as solutions and coefficients

Arithmetica

Collection of 130 problems in solving equations (although only 6 of the original 13 books survive). Introduced algebraic symbolism and Diophantine equations.

How old was Diophantus?

Here lies Diophantus, the wonder behold.
Through art algebraic, the stone tells how old:
'God gave him his boyhood one-sixth of his life,
One twelfth more as youth while whiskers grew rife;
And then yet one-seventh ere marriage begun;
In five years there came a bouncing new son.
Alas, the dear child of master and sage
After attaining half the measure of his father's life
chill fate took him.
After consoling his fate by the science of numbers
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$$\frac{1}{6}L + \frac{1}{12}L + \frac{1}{7}L + 5 + \frac{1}{2}L + 4 = L \Leftrightarrow \frac{3}{28}L = 9 \Leftrightarrow L = 84$$

Marignal notes in *Arithmetica*

Fermat's Last Theorem

If an integer n is greater than 2, then

$$a^n + b^n = c^n$$

has no solutions in non-zero integers a, b, and c.

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> *Pierre de Fermat (1601-1665)*

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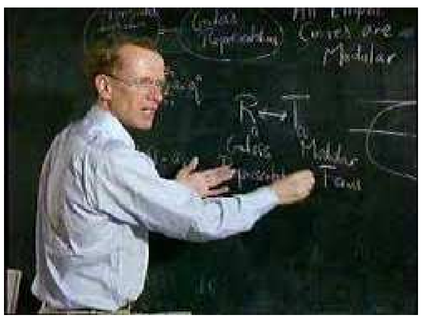

Proof by Andrew Wiles (1994)

Mohamed ibn-Muso al-Khwarizmi (c.790-840)

- Arabian librarian
- lived and worked in Persia
- calculated latitudes and longitudes for 2402 localities as a basis for a world map
- wrote about sundials and the Jewish calendar

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al-Khwarizmi $\equiv$ algorithm, al-jabr $\equiv$ algebra

Science of Reunion and Opposition

al-jabr	reunion	collecting like terms
al-muqâbalah	opposition	cancelling terms

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squares equal to roots	$x^2 = 5x$
squares equal to numbers	$x^2 = 4$
roots equal to numbers	$5x = 15$
squares and roots equal to numbers	$x^2 + 10x = 39$
squares and numbers equal to roots	$x^2 + 21 = 10x$
roots and numbers equal to squares	$3x + 4 = x^2$

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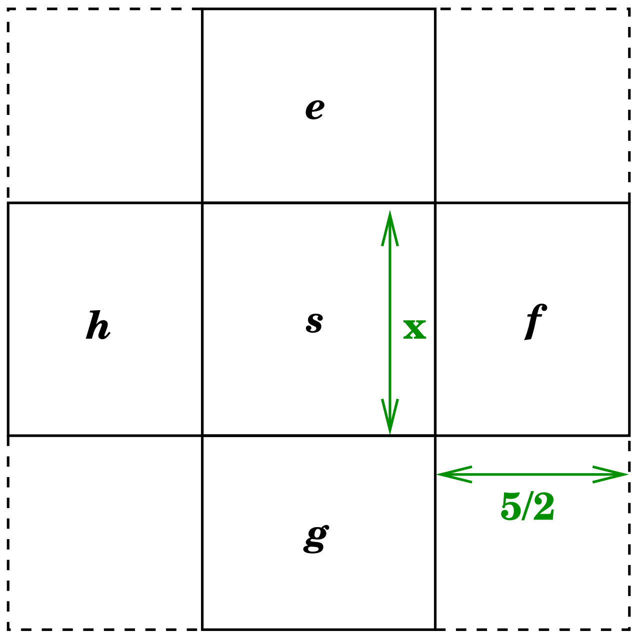
approximations of π:

$$\frac{22}{7}\ (= 3.\overline{142857}), \quad \sqrt{10}\ (\simeq 3.162278), \quad \frac{62832}{20000}\ (= 3.1416)$$

Completing the Square

$$x^2 + 10x = 39$$



Completing the Square

1. x^2 term: centre square s

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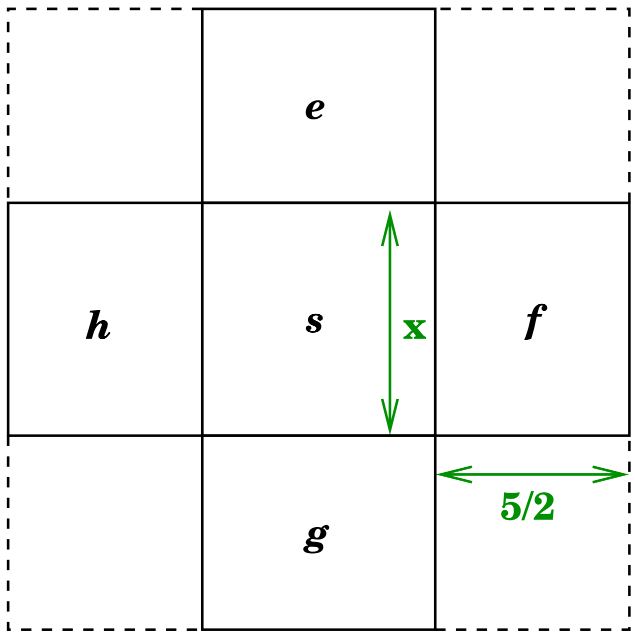

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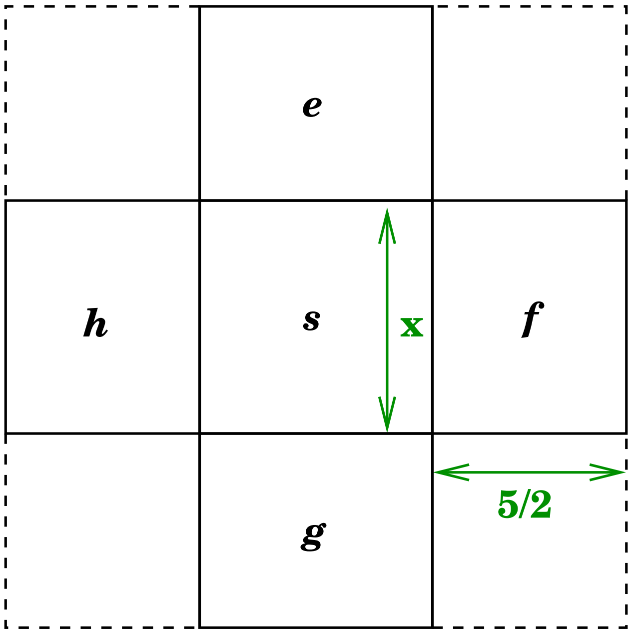

1. x^2 term: centre square s
2. $10x$: add e, f, g, h, each $5/2$ units wide (so area of each is $10x/4$ units)

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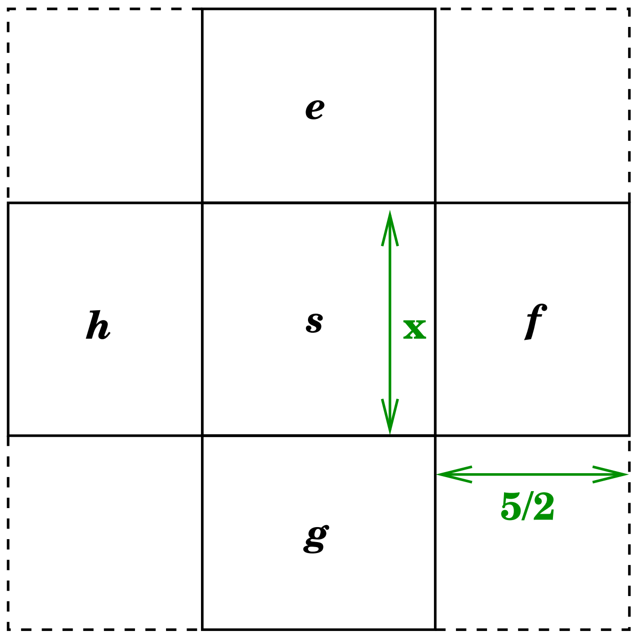

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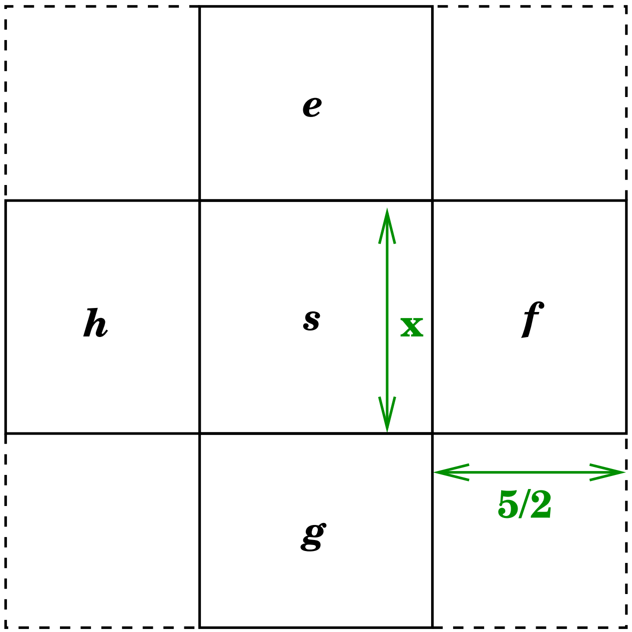

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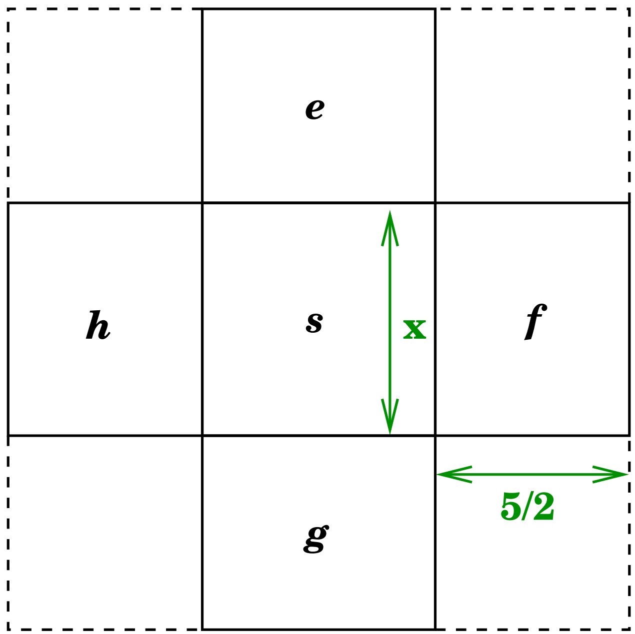

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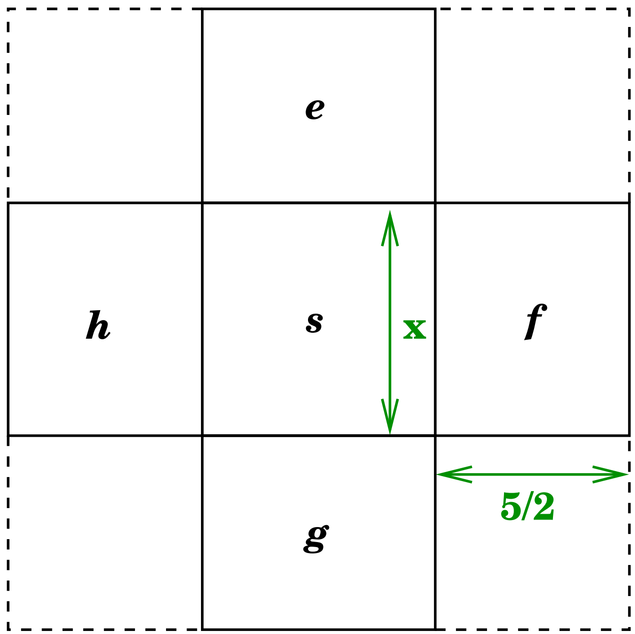

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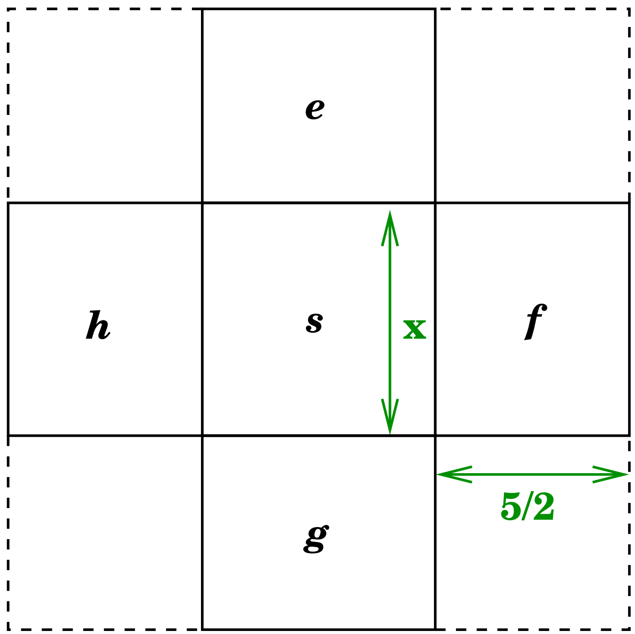

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6. side of large square has length $x + 5 = 8$ units
7. solution is $x = 3$

Using algebra

- PROBLEM: You divide ten into two parts: multiply the one by itself; it will be equal to the other taken eighty-one times.

$$(10 - x)^2 = 81x$$

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- SOLUTION: Ten less thing, multiplied by itself, is a hundred plus a square less twenty things, and this is equal to eighty-one things.

$$x^2 - 20x + 100 = 81x$$

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$$x^2 - 20x + 100 = 81x$$

- Separate the twenty things from a hundred and a square, and add them to eighty-one. It will then be a hundred plus a square, which is equal to a hundred and one roots.

$$x^2 + 100 = 101x$$

Using algebra

- Halve the roots; the moiety is fifty and a half.

$$(x-p)(x-q)=0 \Leftrightarrow \frac{p+q}{2}=\frac{101}{2}=50\frac{1}{2}$$

(and $pq=100$).

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- Multiply this by itself, it is two thousand five hundred and fifty and a quarter.

$$\left(\frac{p+q}{2}\right)^2=2550\frac{1}{4}$$

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$$\left(\frac{p+q}{2}\right)^2=2550\frac{1}{4}$$

- Subtract from this one hundred; the remainder is two thousand four hundred and fifty and a quarter.

$$\left(\frac{p-q}{2}\right)^2=\left(\frac{p+q}{2}\right)^2-pq=2550\frac{1}{4}-100=2450\frac{1}{4}$$

Using algebra

- Extract the root from this; it is forty-nine and a half.

$$\frac{p-q}{2} = \sqrt{\left(\frac{p+q}{2}\right)^2 - pq} = \sqrt{2450\frac{1}{4}} = 49\frac{1}{2}$$

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- Subtract this from the moiety of the roots, which is fifty and a half. There remains one, and this is one of the two parts.

$$q = \left(\frac{p+q}{2}\right) - \left(\frac{p-q}{2}\right) = 50\frac{1}{2} - 49\frac{1}{2} = 1.$$

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Roots of $(10-x)^2 = 81x$ are 1 and 100.

Leonardo Pisano Fibonacci (1170-1250)

- Italian traveller
- originally from Pisa
- brought Arabic maths to Europe
- challenged by Johannes of Palermo
- wrote Liber Abaci

Proved root of $x^3 + 2x^2 + 10x = 20$ is neither an integer nor a fraction, nor the square root of a fraction.

> *And because it was not possible to solve this equation in any other of the above ways, I worked to reduce the solution to an approximation.*

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$$x \simeq 1.3688081075$$

Luca Pacioli (1445-1517)

- Franciscan friar in Sansepolcro, Italy
- the father of accounting
- unpublished treatise on chess illustrated by Leonardo da Vinci

Summa de arithmetica, geometrica, proportioni et proportionalita

Summary of current work in arithmetic, algebra, geometry and trigonometry.

Pacioli's notation

$$\begin{array}{r} 6.p.R.10 \\ 18.m.R.90 \\ \hline 108.m.R.3240.p.R.3240.m.R.90 \\ \text{hoc est } 78. \end{array}$$

$$(6+\sqrt{10})(18-\sqrt{90}) = (108-\sqrt{3240}+\sqrt{3240}-\sqrt{900}) = 78$$

"..., one cannot give general rules except that, sometimes, by trial, ... in some particular cases."

Scipione del Ferro (1465-1526)

- Professor in Bologna, Italy
- solved depressed cubic equations of the form $x^3 + mx = n$ algebraically
- no knowledge of negative numbers
- solution by radicals: find the roots by adding, subtracting, multiplying, dividing and taking roots of the coefficients.
- did not publish any of his work
- passed his greatest discovery on to his student Fior from his deathbed

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" It is often more convenient to possess the ashes of great men than to possess the men themselves during their lifetime."
Jacobi (1804-1851)

Niccolo Fontana Tartaglia (1500-1557)

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- won a public equation-solving contest with Fior

Fior: unknowns and cubes equal to numbers $x^3 + mx = n$

Tartaglia: squares and cubes equal to numbers $x^3 + mx^2 = n$

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- correctly predicted the exact date of his own death: by committing suicide…

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- T was furious and published *New Problems and Inventions* containing malicious personal insults about C.

Cardano-Tartaglia Method for Cubics

"Quando che'l cubo con le cose appresso
Se agguaglia a qualche numero discreto
Trovati dui alte differenti in esso
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When the cube and its things near — $x^3 + mx = n$
Add to a new number, discrete,
Determine two new numbers different — $a^3 - b^3 = n$
By that one; this feat
Will be kept as a rule

Their product always equal, the same, — $a^3 b^3 = \left(\frac{m}{3}\right)^3$
To the cube of a third
Of the number of things named.
Then generally speaking,
The remaining amount
Of the cube roots subtracted — $x = a - b$
Will be your desired count.

Solving $x^3 + mx = n$

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- cubic has solution $x = a - b$

Solving $x^3 + mx = n$ cont.

$$x = \left(\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}\right)^{\frac{1}{3}} - \left(-\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}\right)^{\frac{1}{3}}$$

- Case 1

$$x^3 + mx = n$$

$$x^3 + 6x = 20 \Rightarrow m = 6, n = 20$$

$$x = \left(10 + \sqrt{108}\right)^{\frac{1}{3}} - \left(-10 + \sqrt{108}\right)^{\frac{1}{3}} = 2$$

Solving $x^3 + mx = n$ cont.

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- Case 2

$$x^3 = mx + n$$

(use $x = a + b$)

$$x^3 = 15x + 4 \Rightarrow m = 15, n = 4$$

$$x = \left(2 + \sqrt{-121}\right)^{\frac{1}{3}} - \left(2 - \sqrt{-121}\right)^{\frac{1}{3}}$$

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PROBLEM: $x = 4$ is a solution, equation not insoluble!

Ferrari's Method for Quartics

depressed quartic: $x^4 + px^2 + qx + r = 0$

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- add in any additional term y:

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\begin{aligned}
(x^2 + p + y)^2 &= (x^2 + p)^2 + 2y(x^2 + p) + y^2 \\
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&= (p + 2y)x^2 - qx + (p^2 - r + 2py + y^2)
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$$q^2 - 4(p + 2y)(p^2 - r + 2py + y^2) = 0$$

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- taking square root of both sides gives a quadratic equation for x

Niels Henrik Abel(1802-1829)

- Norwegian pauper and invalid
- published papers in the first ever mathematical journal
- results rejected by the French Academy as illegible
- died of tuberculosis
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On algebraic equations in which the impossibility of solving the general equation of the fifth degree is demonstrated

Quintics are insoluble via method of radicals.

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> "Abel has left mathematicians enough to keep them busy for 500 years."
> *Hermite (1822-1901)*

Evariste Galois (1811-1832)

- French student and soldier
- Cauchy lost his French Academy paper, Fourier died before receiving it
- imprisoned for treason
- persecution complex
- killed in a duel aged 20

Researches on the algebraic solution of equations

Showed when a general equation could be solved by radicals.

Galois theory of groups

Sir Isaac Newton (1642-1727)

- born in Grantham
- studied at Cambridge University
- sent home due to the Plague in 1665
- Professor at Cambridge
- Warden of the Royal Mint

De analysi per aequationes numero terminorum infinitas

Iterative method for finding approximate roots of algebraic equations.

Johann Carl Friedrich Gauss (1777-1855)

- Numerical methods for solving equations
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"I have had my results for a long time: but I do not yet know how to arrive at them."
Gauss

Some final thoughts . . .

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"An expert is someone who knows some of the worst mistakes that can be made in his subject, and how to avoid them."
Werner Karl Heisenberg (1901-1976)

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Thanks for listening!