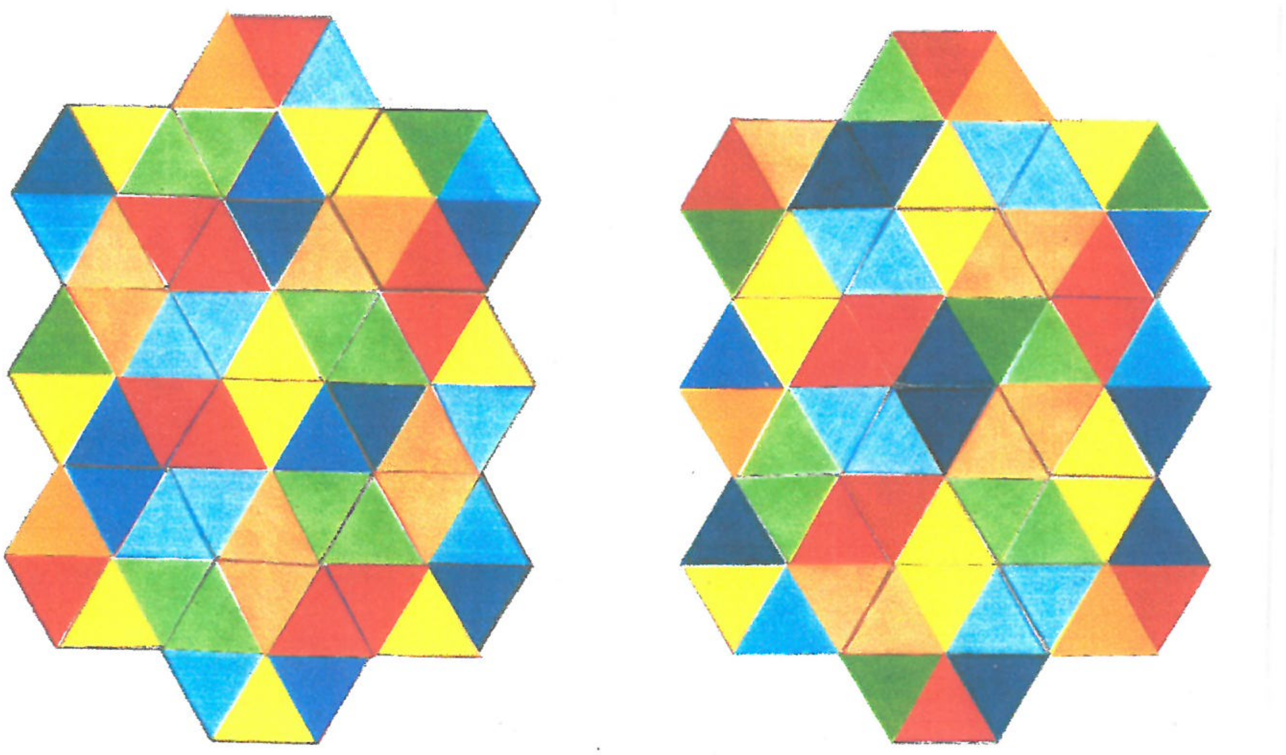
MAthematical PiE

Autumn 2020 **Notes for . . . . . No 211**

Coloured Hexagons

Don't forget that the colour of a blank triangle can not only be fixed by an adjoining triangle, a colour can also be ruled out by the colours already in an adjoining hexagon.



Dogs and People

No comment needed.

Fireworks.

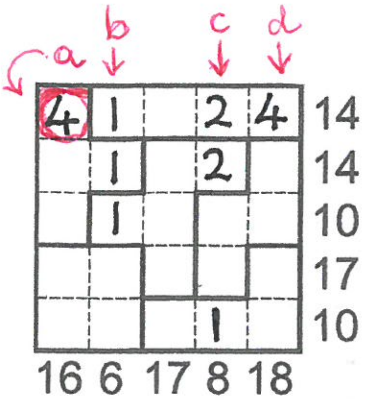
The first customer buys 15 + 18 = 33 fireworks and the second 16 + 19 + 31 = 66. The box of 20 rockets is left.

Graphiti?

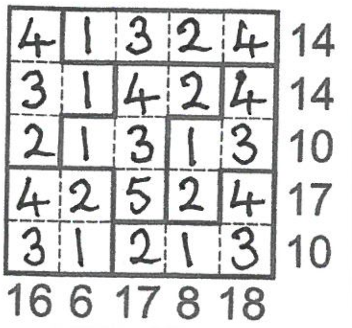
Suppose that each brick has length $\boldsymbol{l}$ and height $\boldsymbol{h}$. Then AB = $4\boldsymbol{l}$ and the height of ABC is $2\boldsymbol{h}$. So the area of ABC is half of $4\boldsymbol{l} \times 2\boldsymbol{h} = 4\boldsymbol{lh}$ i.e. four bricks or 400cm^2.

Polyominumbered

The main consideration is the potential contribution that a polyomino can make to a total. For example, if a tetromino has two cells in a particular row, the smallest contribution it can make to the total is 1 + 2 = 3, or the maximum would be 4 + 3 = 7. Keep an eye on when some of the cells of the shape have already been found. Below are the first few steps a,b,c,d based on these tactics:

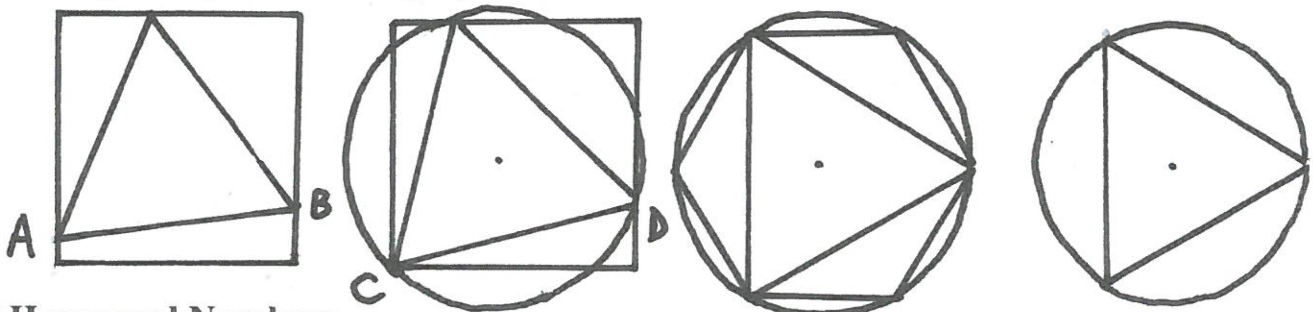


The final result is:



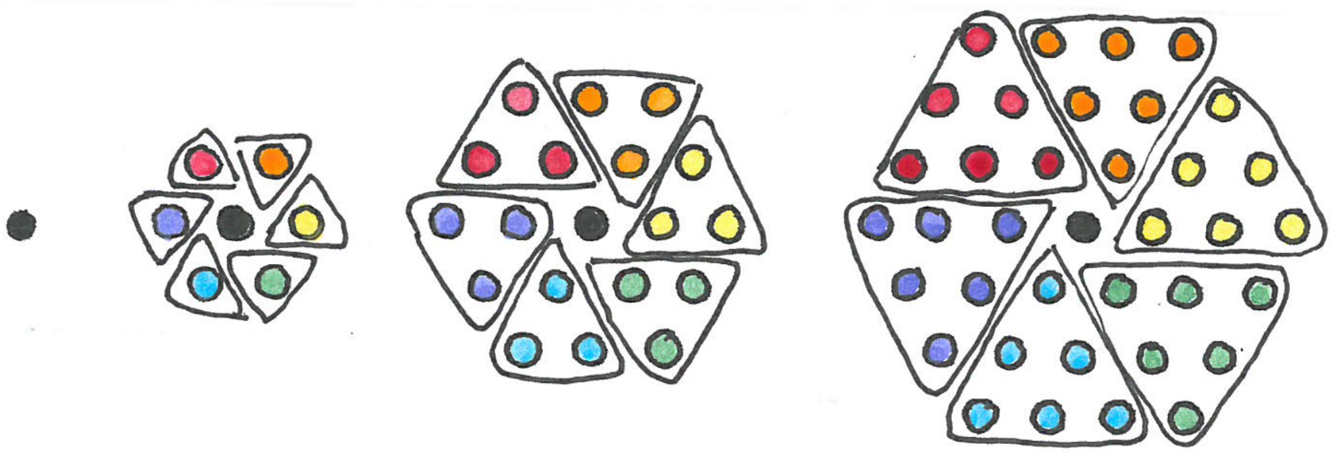
Reason and Intuition

Any other equilateral triangle drawn in the square will be smaller: In the example below, AB is smaller than CD since it is at less of an angle to the horizontal base. Drawing circles around the best triangles in the square and hexagon, reason tells us that the circle around the hexagon is larger than the final circle because it has a greater area: however the decision that the middle circle is larger than the first is more intuitive (but in fact correct). It is difficult to tell on this scale, but in larger diagrams it is slightly more obvious.



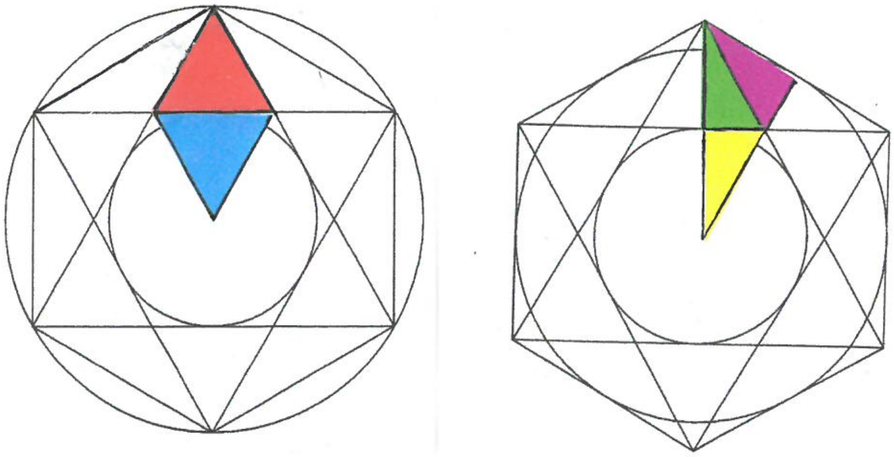
Centred Hexagonal Numbers

The first diagrams show the basis for an iterative formula for c(n), namely $c(n-1) + 6(n-1)$. The second set show $c(n) = 3n(n-1) + 1$, the third set show $c(n) = n^2 + 2(n-1)^2 + (n-1)$, and the fourth set show $c(n) = 3n^2 - 3n + 1$. A little algebra will soon reveal their equivalence. The fourth set could also be interpreted in 3D as representing three faces of a cube. If each dot were itself a cube, we would have a 'large' cube with a slightly smaller cube removed from the unseen corner. So $c(n) = n^3 - (n-1)^3$. Alternatively, fitting six triangular numbers around a central dot gives $c(n) = 6t(n-1) + 1$. There are many other possibilities.



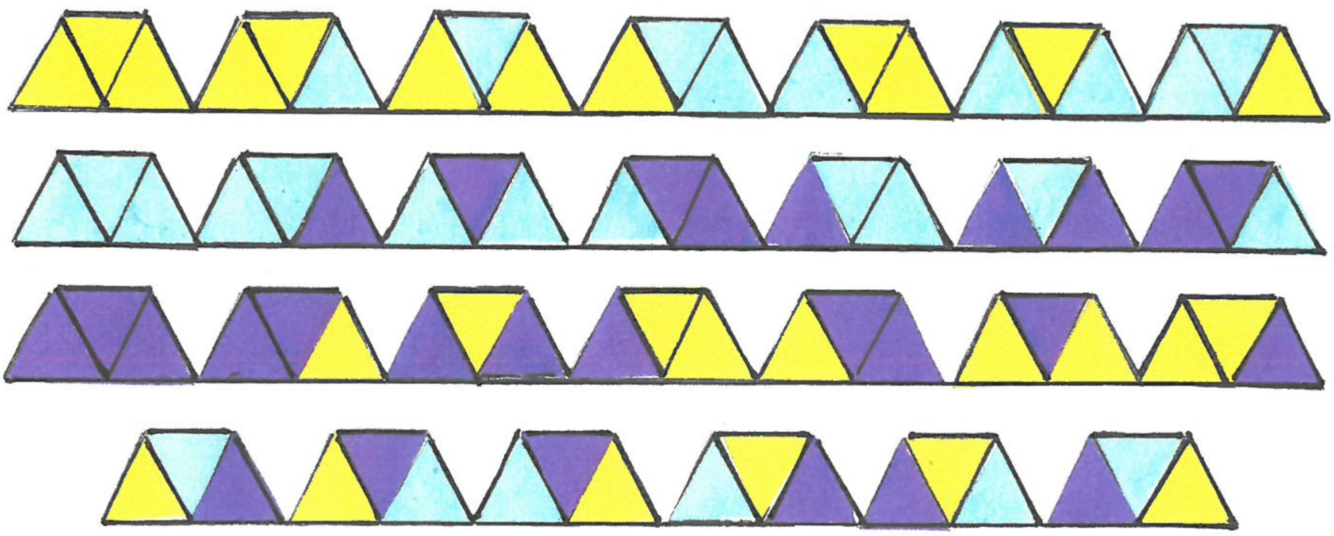
A Pop–up Dodecahedron
This idea may have been around for a long time but it will still be new to some.

Hexagons and Circles
Comparison of the triangles drawn below shows that the first small circle has a radius half that of the larger one, and the second has an area one-third of the larger one:

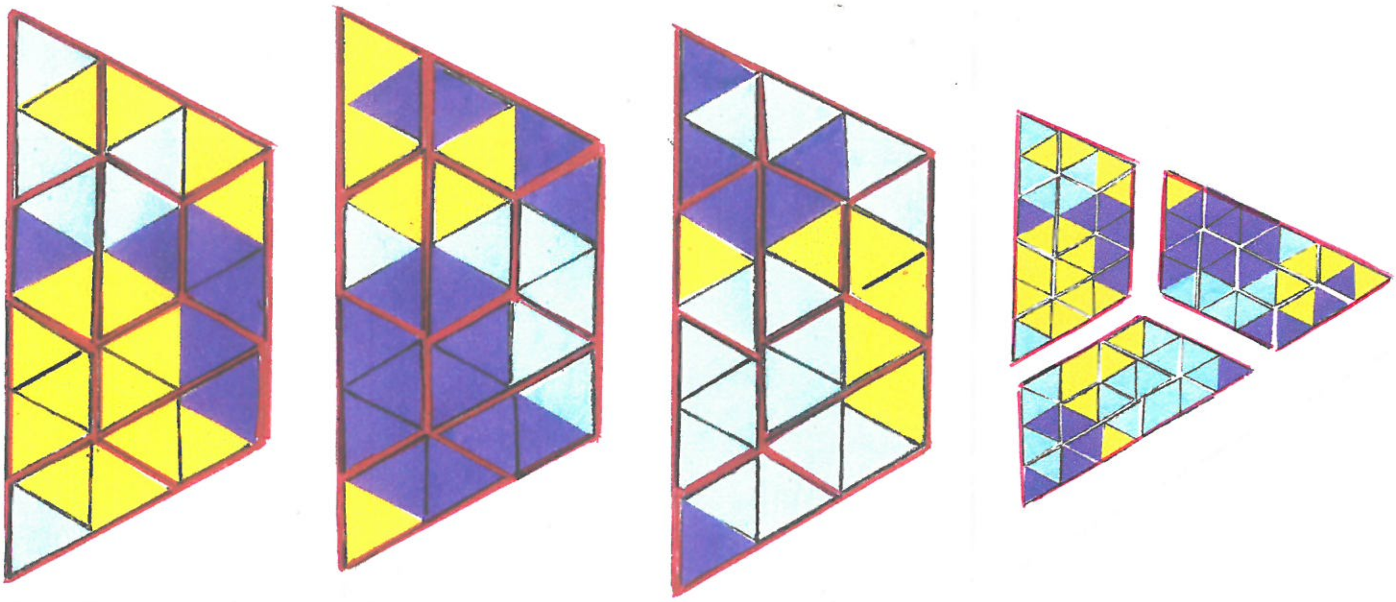


The Biggest Number
To make the longest number possible we need to visit all nine cells, which forces us to start either at a corner or in the centre. The best is 594836271.

Tried Trapeziums
When colouring, such as with McMahon triangles or squares, we do not discard mirror-images (since the 'turned over' tile would not be coloured on the back): so the number of colourings is simply 3 x 3 x 3 = 27. Here are the 27 trapeziums in a 'reasonably logical' array:



It is possible to fit the pieces together on an edge-match basis in many ways. Only practical cutting-out and playing (in my case, sometimes for hours!) will produce results. This kind of trapezium is an example of a 'rep-tile' and so 4, 9, 16 or 25 of them could be used to form 'enlarged' versions of the original. It is interesting to see if these can be made with edge matching, and even more interesting when subsets can be used to make congruent shapes with matching patterns. Overleaf is a set of three 'trebled' trapeziums:



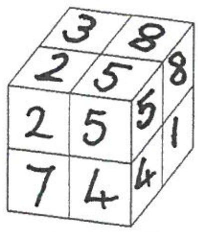
Since 27 trapeziums are formed from 81 triangles, the idea of a large 'nine times' triangle would seem to be viable. In fact the three pattern-matching trapeziums will easily fit together to make one edge-matching triangle: as shown on the right above.

The Three Presents
If the prices are **a, b, c,** we might have, say, **b** = 2**a** but we cannot be sure whether **a** = 3**c**, **c** = 3**a**, **b** = 3**c** or **c** = 3**b**. All we can do, I think, is try each in turn. The third option gives the solution with the three prices being £1.86, £3.72 and £1.24.

The Tiny Square
The large square has an area of $24^2 + 10^2 = 676$, so its edges measure 26cm.

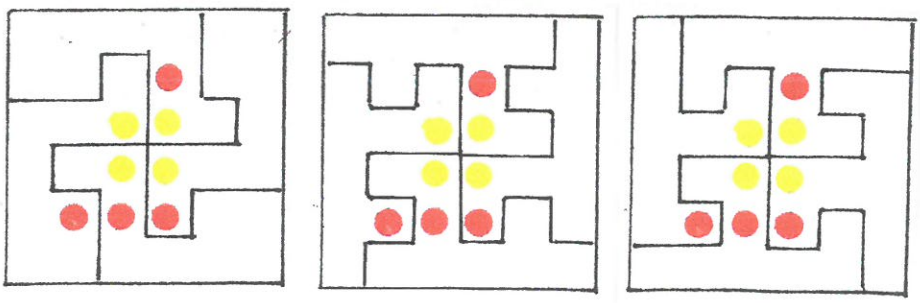
On All the Faces
Each cube contributes three faces to the large cube, so the overall total of the numbers on its six faces must be 3 x (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8) = 108.
Hence each face must total 18:

Top 3 + 8 + 2 + 5 =18
Front 2 + 5 + 7 + 4 =18
Right 5 + 8 + 4 + 1 = 18



Left 2 + 3 + 7 + 6 = 18
Back 3 + 8 + 6 + 1 =18
Bottom 7 + 4 + 6 + 1 = 18

Four Identical Shapes
There are three solutions:



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