Reasoning as a mathematical habit of mind

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What is Teaching?

Creating a
common experience
to reflect on
and so bring about learning.

True or false?

$39 \times 46 = 39 \times 45 + 46$

$39 \times 46 = 39 \times 45 + 39$

Much mathematical reasoning is independent of arithmetical fluency.

True or false?

$39 \times 46 = 39 \times 45 + 39$

$3 \times 5 = 3 \times 4 + 3$

$326 \times 18 = 326 \times 17 + 326$

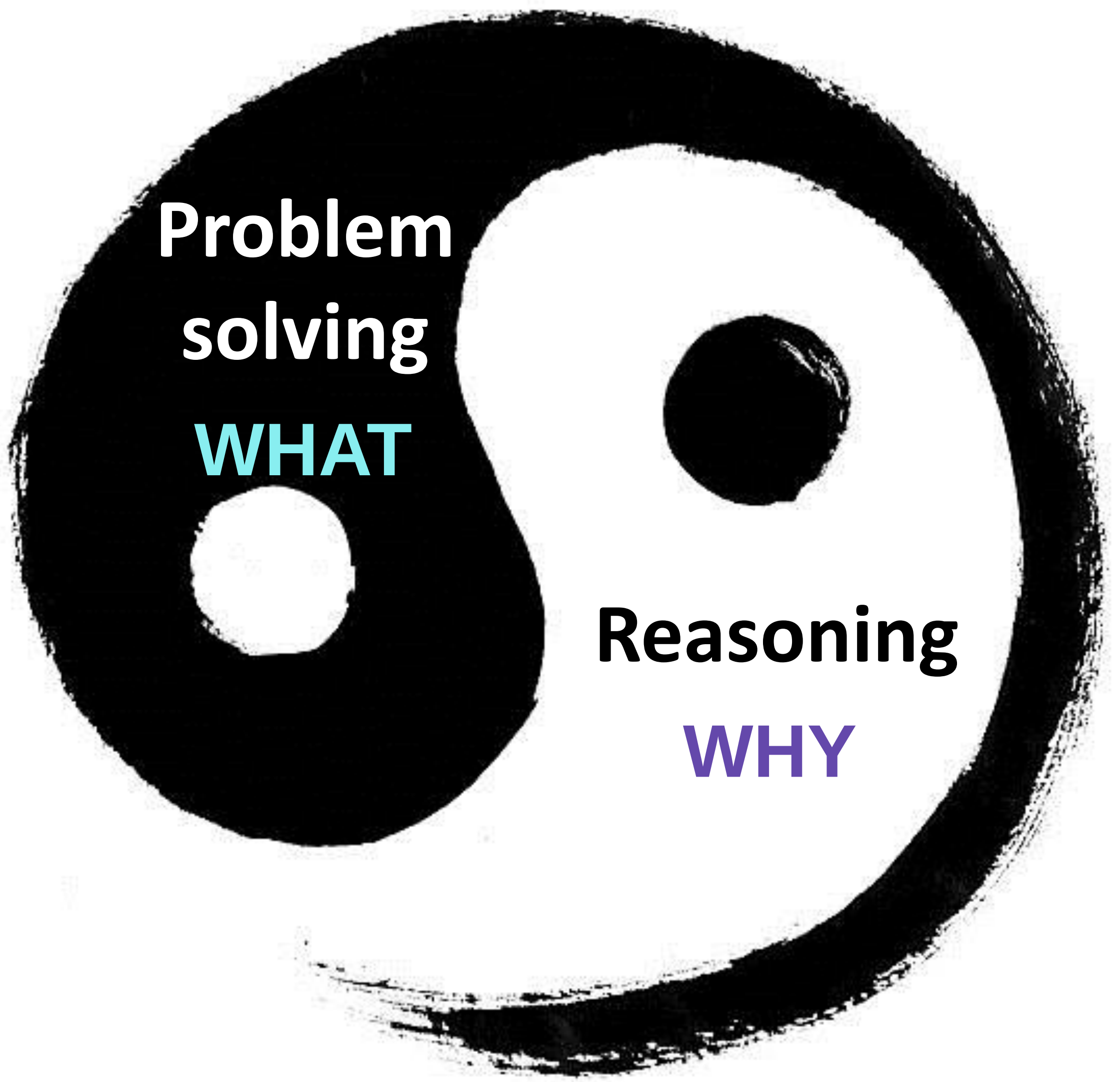
21

$$5{,}542 \div 17 = 326$$

Explain how you can use this fact to find the answer to **18 × 326**

1 mark

Reasoning – What is it?


Problem solving
WHAT
Reasoning
WHY

Additive reasoning

Statistical reasoning

Multiplicative reasoning

Proportional reasoning

Probabilistic reasoning

Approximate reasoning

Geometric reasoning - positional

Geometric reasoning - axiomatic

Types of mathematical reasoning

Deductive

- Draw conclusions based on known facts

Inductive

- Generate examples, ‘sniff out’ patterns

Abductive

- Conjecture as to why?

Analogical

- Identify and draw on similarities with things already understood

Relational

- Identify connections between quantities or shapes that do not require numbers or measures

Deductive

Magic squares

	3	
2		**1**

Sum of 9

	2	
1		**5**

Sum of 6

	4	
2		**2**

Sum of 8

Given three numbers in the these positions,
and the sum, when can you create a magic square?

Source: Arcavi (1994)

Inductive

High-low difference

Write down three digits

1. Arrange the digits from largest to smallest
2. Arrange the digits from smallest to largest
3. Find the difference between the two numbers.

Repeat with the answer

What do you notice?

What do you wonder?

Abductive

Potential to reason

5 x 6 = 30

7 x 8 = 56

4 x 5 = 20

6 x 7 = 42

Potential to reason

5 x 6 = 30

7 x 8 = 56

4 x 5 = 20

6 x 7 = 42

4 x 5 = 20

5 x 6 = 30

6 x 7 = 42

7 x 8 = 56

What do you notice?

What do you wonder?

Analogical

Analogical

A mathematical answer to 27 divided by 6 is 4 remainder 3.

Make up a real world problem that involves 27 divided by 6 but where it makes sense to round the answer up to 5.

Relational

Tea party

On Saturday some friends came to tea.

We shared a packet of biscuits.

On Sunday I had another tea party.

More friends came round and we shared the same number of biscuits.

Did each person on Sunday eat more, less or the same as each did on Saturday?

Source: Susan Lamon (2005)

Biscuits Friends	Fewer	Same	More
Fewer			
Same			
More		Less	

Biscuits Friends	Fewer	Same	More
Fewer		More	More
Same	Less	Same	More
More	Less	Less	

Mathematical reasoning

- In 7 out of 9 cases you can find the answer **without knowing any quantities**

- Relational reasoning is independent of arithmetical ability

Biscuits
Friends

$\frac{5}{8}$ $\frac{7}{8}$ $\frac{5}{6}$ $\frac{5}{8}$

$\frac{5}{9}$ $\frac{7}{8}$ $\frac{5}{6}$ $\frac{4}{7}$

Relational

Sue and Julie are cycling equally fast around a track.

Sue started first.

When Sue had cycled 9 laps,
Julie had cycled 3 laps.

When Julie completed 15 laps, how many laps had Sue cycled?

Types of mathematical reasoning

Deductive

Inductive

Abductive

Analogical

Relational

- Children, from an early age, engage with all these types of reasoning.
- They are not easy to separate.
- We can help children learn more about mathematical reasoning.

Research Report DCSF-RR118

Development of Maths Capabilities and Confidence in Primary School

Terezinha Nunes, Peter Bryant, Kathy Sylva and Rossana Barros
Department of Education,
University of Oxford
In collaboration with ALSPAC, University of Bristol

DCSF-RR118

department for
children, schools and families

Mathematical reasoning, even more so than children's knowledge of arithmetic, is important for children's later achievement in mathematics.

Nunes et al. DSFC RR-118

Reasoning – Why is it difficult to enact?

National Curriculum Aim

The national curriculum for mathematics aims to ensure that all pupils:

- **reason mathematically** by following a line of enquiry, conjecturing relationships and generalisations, and developing an argument, justification or proof using mathematical language

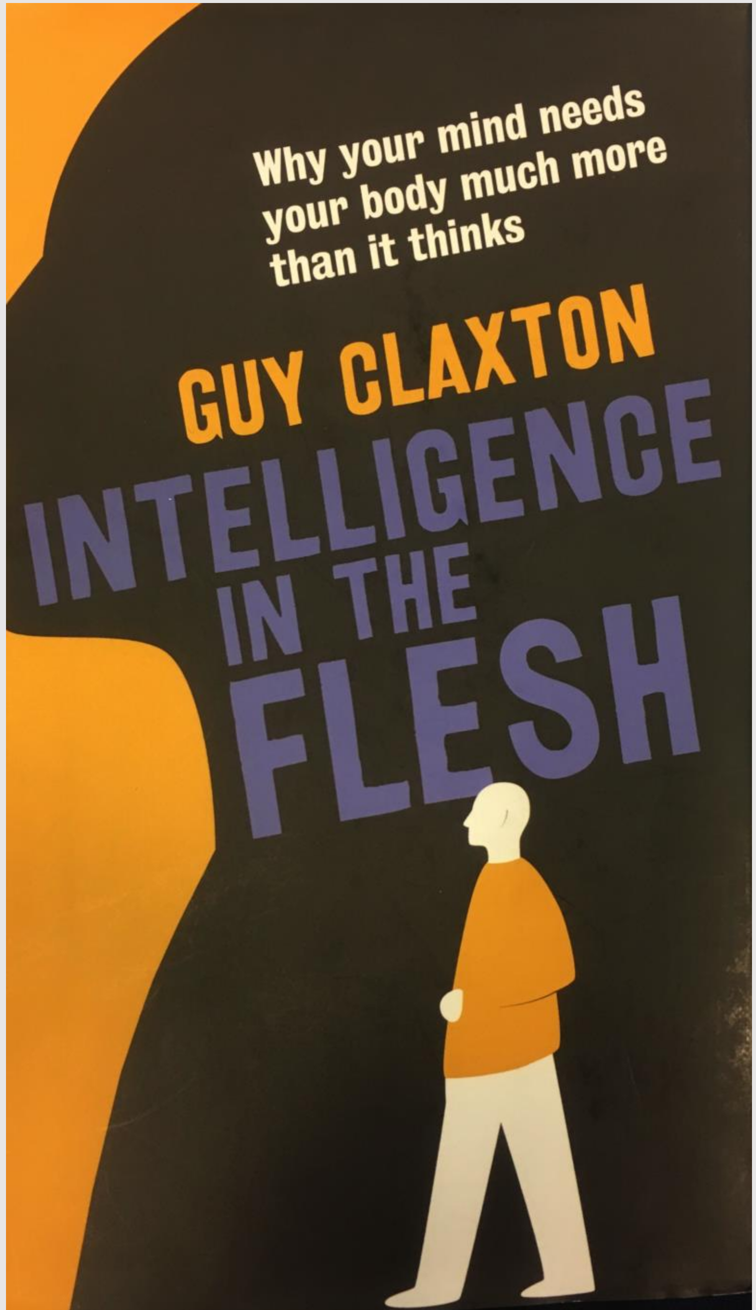
Myths

It’s a ‘big production number’

Fluency and problem solving have priority

It’s only for the ….

It’s ‘developmental’


Why your mind needs
your body much more
than it thinks
GUY CLAXTON
INTELLIGENCE
IN THE
FLESH

One of the major errors of twentieth-century psychology was to suppose that there are childish ways of knowing which are outgrown, and ought to be transcended, as one grows up. The childish ones are the bodily ones, and are to do with concrete action and experience. The grown-up ones are abstract, logical and propositional. But it is a Cartesian mistake to think that, once you have mastered logic you don’t need the body anymore. … We should think of the developing mind as a tree that grows new branches, not as a spaceship whose booster rockets fall away for ever one they have done their job and are spent.

Claxton, 2015, p 165-66

How we reason mathematically is does not change substantially over the years.

What we reason about mathematically can change dramatically over the years.

Challenges

'Push back'

Planning

Teaching demands

Maintaining mathematical challenge

Group of teachers planned tasks with researchers

Focus on reasoning and problem solving

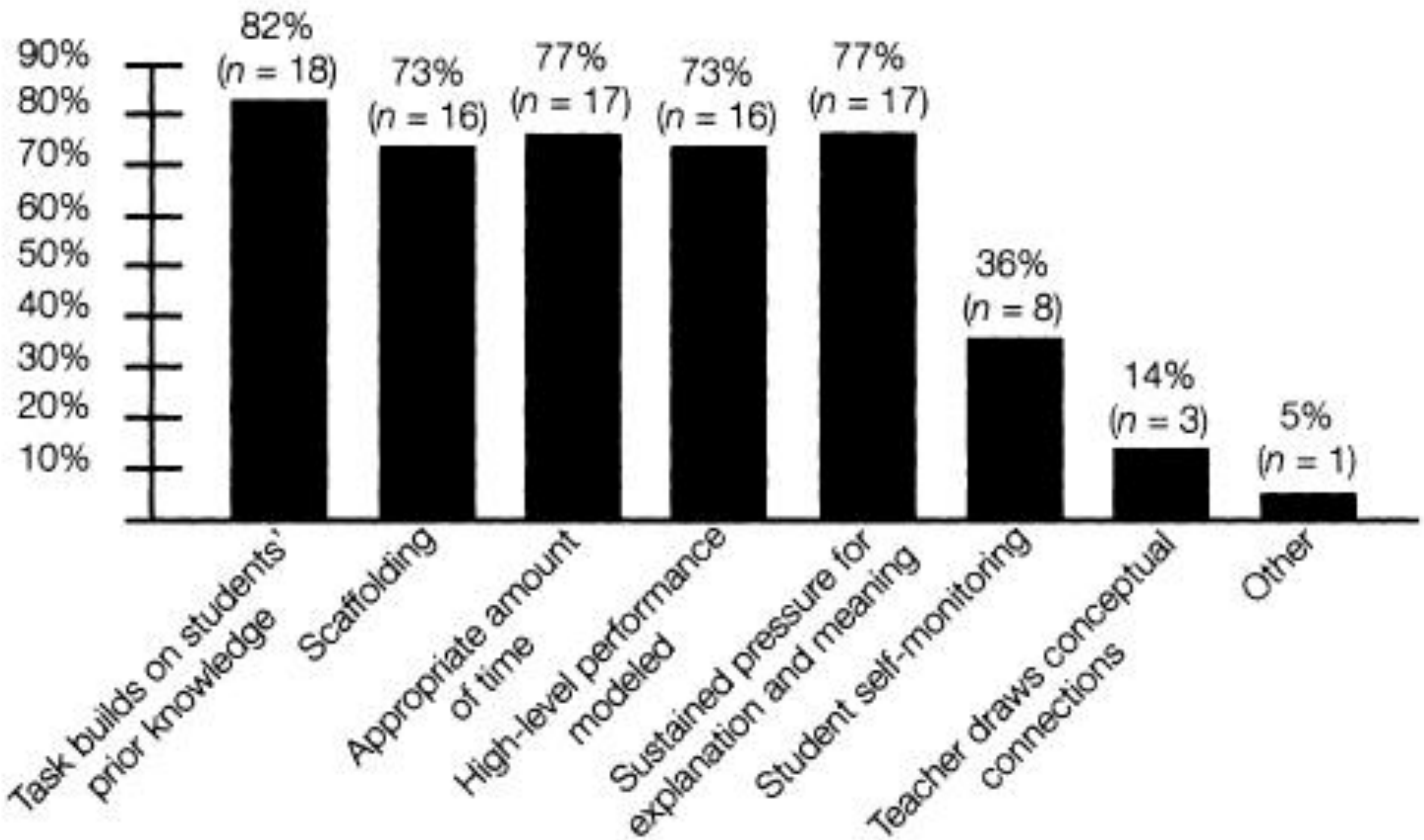
68 lessons

Maintaining mathematical challenge

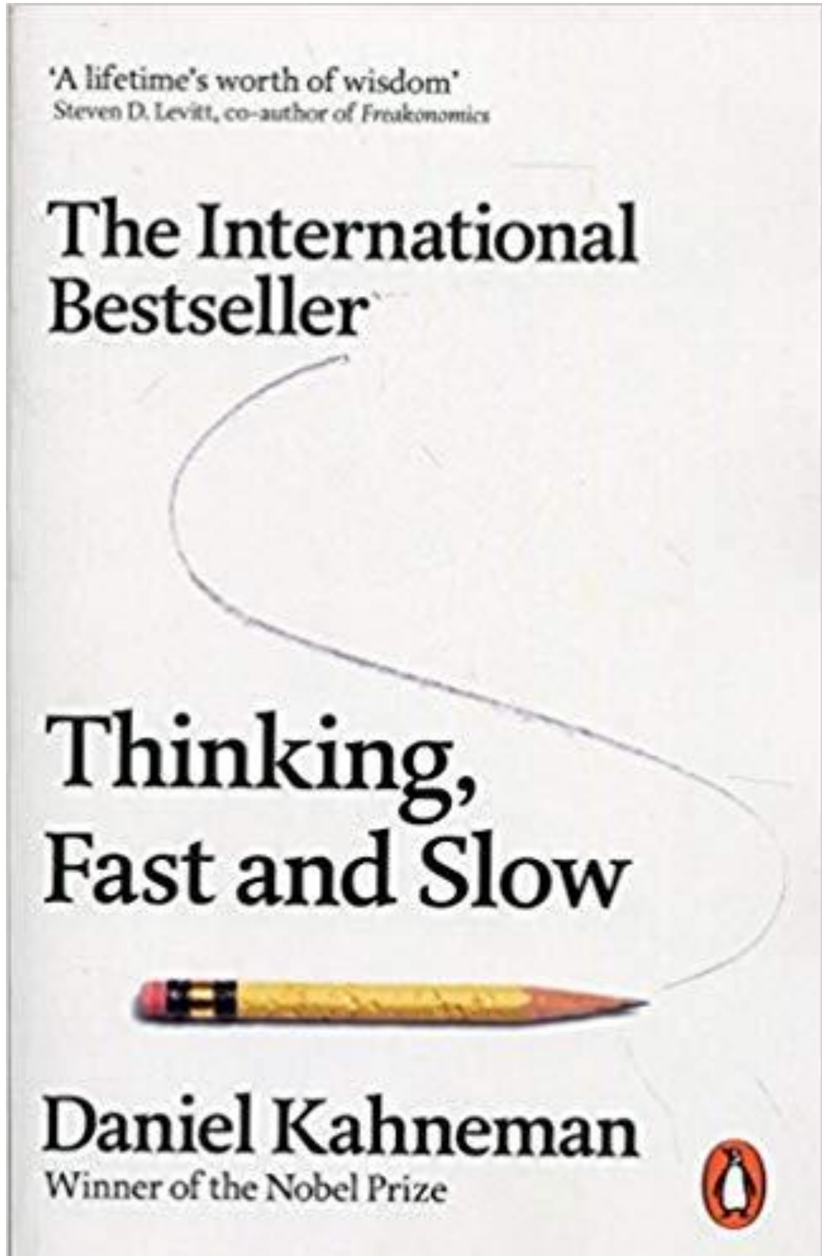
When taught two thirds of activities ‘declined’ into

- Procedures
- Non-systematic
- Non mathematical

Maintain high level



Source: Henningson & Stein (1997)


'A lifetime's worth of wisdom'
Steven D. Levitt, co-author of Freakonomics
The International Bestseller
Thinking, Fast and Slow
Daniel Kahneman
Winner of the Nobel Prize

Moving from Fast to Slow thinking …

… results in a slight, but noticeable, feeling of depression

Objects of learning

Indirect

- Fluency
- Problem solving
- Reasoning

AIMS

Direct

- Fractions
- Multiplication
- 3-D shapes

CONTENT

Teaching demands

Requires making sense of learners' methods

AND

aligning these with accepted mathematical knowledge.

TALK central to both of these

Discussion

or

Dialogue?

Discussion

David Bohm likened discussion to *throwing* opinions back and forth in an attempt to convince each other of the rightness of a particular position.

The whole view is becomes fragmented and shattered into many pieces.

Dialogue

Dialogue requires willingness and skill to engage with minds, ideas and ways of thinking other than our own; it involves the ability to question, listen, reflect, reason, explain, speculate and explore ideas; to analyse problems, frame hypotheses, and develop solutions; …

Dialogue within the classroom lays the foundations not just of successful learning, but also of social cohesion, active citizenship and the good society.

Robin Alexander

Dialogic talk

Speaking AND listening

Building on ideas, not reporting

- Repeat
- Re-voice
- Rephrase
- Build on
- Comment on

Structure

Freedom

Experienced teachers do two apparently contradictory things: They use more structures, and yet they improvise more. … The challenge facing every teacher and every school is to find the balance of creativity and structure that will optimise student learning.

Sawyer

Five orchestrating practices

Anticipating

Monitoring

Selecting

Sequencing

Making connections between student responses

(Source: Stein, Engle, Smith, Huges, 2009)

Reasoning – A mathematical habit of mind

Mathematical power

Mathematical power is best described by a set of *habits of mind*. People with mathematical power perform thought experiments; tinker with real and imagined machines; invent things; look for invariants (patterns); make reasonable conjectures; describe things both casually and formally (and play other language games); think about methods, strategies, algorithms, and processes; visualize things (even when the "things" are not inherently visual); seek to explain *why* things are as they seem them; and argue passionately about intellectual phenomena.

Cuoco, Goldenberg and Mark (1996)

Reasoning chains

Addition and subtraction

5 + 5

5 + 6

10 + 10

10 + 9

Addition and subtraction

62 + 10

62 + 30

62 + 29

54 + 19

Addition and subtraction

62 – 10

62 – 30

62 – 29

54 – 19

Multiplication and division

6 x 10

6 x 40

6 x 39

Multiplication and division

$21 \div 7$

$70 \div 7$

$91 \div 7$

$270 \div 27$

$297 \div 27$

Multiplication and division

$160 \div 16$

$320 \div 16$

$320 \div 32$

True or false?

Statements or questions

If you add four even numbers, the answer is a multiple of 4.

If you add four even numbers, is the answer always a multiple of 4?

Responding to statements is more productive of reasoning than responding to questions.

True or false?

$20 + 8 = 10 + 18$

$20 + 8 = 19 + 9$

$80 + 8 = 60 + 18$

$8 + 90 = 40 + 50 + 8$

True or false

$37 + 56 - 56 = 37$

$458 + 347 - 347 = 458$

$458 + 347 - 347 = 347$

Conjecture

If you add a number and then take the same number away, you have the number you started with.

True or false?

56 – 38 = 56 – 37 – 1

56 – 38 = 56 – 39 + 1

56 – 38 = 56 – 36 – t

True or false?

$64 \div 14 = 32 \div 28$

$64 \div 14 = 32 \div 7$

$42 \div 16 = 84 \div 32$

True or false?

A semi circle is always half a circle

Half a circle is always a semi-circle

ALWAYS …

… needs to be given a special role in mathematical dialogue

- Do you think that will always be true for …..?

Open number sentences

Open number sentences

64 + 14 = [   ] + 64

64 + [ ] = 18 + 64

64 + [ ] = 18 + 62

Open number sentences

647 – 285 = [ ] – 300

671 – 285 = 640 – [ ]

[ ] – 285 = 640 – 285

Open number sentences

3 x 5 = 5 x [   ]

3 x 5 = 3 x 4 + [   ]

3 x [ ] = 3 x 5 + 3

Open number sentences

70 ÷ [   ] = 7

[   ] ÷ 2 = 35

[   ] x 3.5 = 7

3.5 x 20 = [   ]

70 ÷ [   ] = 3.5

18 Write the missing number.

$$70 \div \boxed{} = 3.5$$

Fill in the blanks

____ is 6 more than ____

_____ is _____ plus 9

7 is _____ minus ______

Create 3 different answers for each.

Fill in the blanks

_____ is 3/5 of _____

8 is ______ (fraction) of ______

______ is ______ (fraction) of 24

Create 3 different answers for each.

Encouraging reasoning requires a shift

From

- Answers as the product

To

- Answers as part of the product

A shift from asking

How do I help learners get the answer to this problem?

To

What mathematics are they expected to learn from working on this problem?

John Dewey – Business of education

“While it is not the business of education … to teach every possible item of information, it is its business to cultivate deep-seated and effective habits of discriminating tested beliefs from mere assertions, guesses, and opinions.”

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