Models of multiplication: Unlock the power

Dietmar Küchemann and **Jeremy Hodgen** explore different models of multiplicative thinking

Introduction

In this article, we look at some models that can be used in the classroom to bring out the structure of multiplication. We discuss both the potential power of such models and the difficulties than can arise in unlocking that power.

The task in Figure 1 comes from a lesson (*Changing expressions*) on multiplication, that we developed recently as part of the work of the ICCAMS project*. Although designed for Year 7 students, this task would work equally well with many Year 5 and 6 classes. Before you read further, please have a go at the task and, more importantly, think of a way, other than simply performing the calculations, to justify your choice of expression.

The purpose of the task is to get a sense of what the numbers in a multiplicative expression are 'doing', by seeing what happens when the numbers are varied slightly. The task is quite demanding (deliberately so) and in the lesson notes we suggest that teachers do not try to resolve it immediately with their class. Instead, we suggest they switch to comparing just A and B, and then just A and C, before returning to the full task. Thus they might ask their class, "Tell me, *without calculating*, how much bigger than A is B?".

A teacher who was trialling the lesson for us with her mixed attainment Year 7 class, gave the initial task to the students on a worksheet. She found that 8 of the 23 students correctly wrote that expression B would be larger, but none gave a comprehensive reason. Three of these 8 students gave reasons that focussed on just one number from each expression, for example "... because the number you're timesing by has to be smaller". The other 5 students referred to 'estimating' or gave no reason at all. Ten students wrote that expressions B and C would be equal. Of these, 6 gave a 'compensation' reason such as the one in Figure 2. Two students opted for C and 3 gave no response.

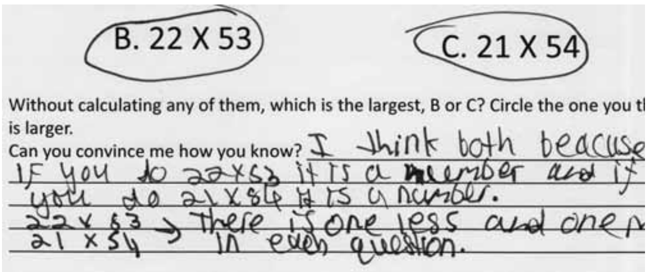

Figure 2: A compensation argument for claiming expressions B and C are equal.

One can argue that saying B and C are equal is not an unreasonable *initial* response, because students may *initially* generalise from their experiences of addition. But why did none of these ten students appear to have second thoughts and examine the situation further? In our development work, we have undertaken numerous interviews with small groups of Year 7 and 8 students. These have often been very stimulating experiences, for the students

Changing expressions

Look at expression A ↓.

Imagine we add 1 to one of the numbers, so we get expression B ↓ or expression C ↓.

Which is larger, B or C?

A. 21×53 B. 22×53 C. 21×54

Figure 1: Introductory task for an ICCAMS lesson examining the structure of multiplication.

as well as us, with students often being thoroughly engaged for periods of nearly an hour. However, we have gained the strong impression from these interviews that many students see mathematics as being essentially about procedures. Thus, when we asked ‘Why?’ questions, we would commonly get ‘How’ answers.

It is possible, therefore, that the notion of making *sense* of a mathematical situation is not the first thing that comes to mind to many students. Rather it is, ‘How do I work out the answer?’. A way to give meaning to a mathematical situation, especially something quite abstract like the expression 21×53 is to *model* the situation in a more concrete way. We decided to make this the focus of the *Changing expressions* lesson, by suggesting that teachers ask students to write stories for the various expressions and to draw diagrams.

Using stories

Writing a story for a multiplication expression is not that straightforward. It is easy enough for an addition expression e.g. for 9 + 3, one might come up with a simple story like “Aysha had 9 apples and Ben had 3 apples. How many did they have altogether?” There are several ways of interpreting multiplication (e.g. Anghileri, 1989) of which the most familiar are probably repeated addition and equal grouping. For these interpretations, the numbers cannot both stand for apples: it does not really make sense to multiply 9 apples and 3 apples, which is not to say that some students will not try, as in the example in Figure 3 (Brown & Küchemann, 1977)**. In the *Changing expressions* lesson, we found that, with a degree of help from their teacher, classes would gradually compile a repertoire of multiplication stories, as in Figure 4. These could then be drawn on, in this and subsequent lessons, to give meaning to the expressions.

Consider, for example, this mini-story for expression A: *‘21 packets of biscuits with 53 inside.’* If one wanted to compare A with B, one could do so by comparing the story with this one: *‘22 packets of biscuits with 53 inside.’* It is

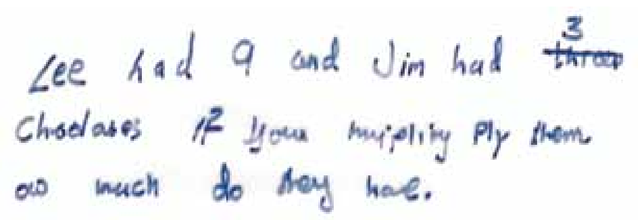

Figure 3: An attempt to write a story for 9×3.

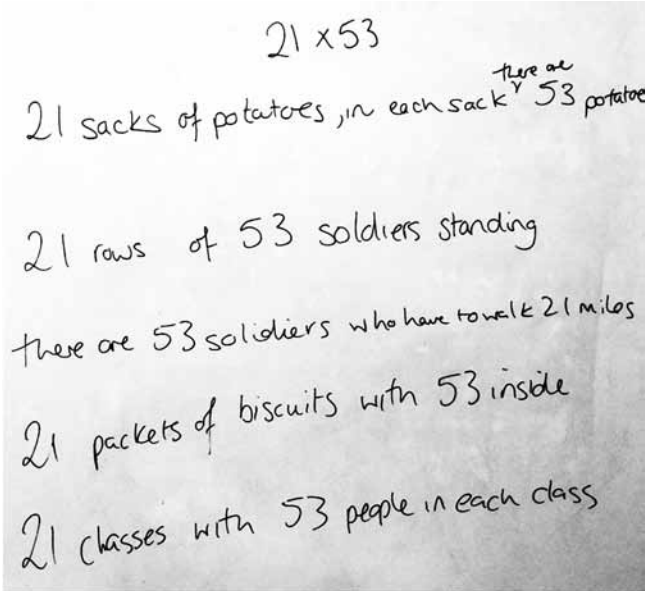

Figure 4: A compilation of viable stories for 21×53.

relatively straightforward to see that this second story involves one extra packet of 53 biscuits, so 22×53 is 53 more than 21×53. Similarly, we can modify the story to fit C: *‘21 packets of biscuits with 54 inside.’* It might not be quite as easy to compare this story to A. Nonetheless, it is likely that some students will see that, compared to A, we have the same number of packets, but with one extra biscuit in each, making 21 extra biscuits altogether.

Using diagrams

As well as writing stories to model the multiplicative expressions, we suggested in the lesson notes that teachers should also use visual representations to model the expressions. In an early draft of the lesson notes, we suggested to teachers that they should focus on the array – as in Figure 5, which shows an array for 21×53. We anticipated that an image like this would be very powerful, since it would seem very clear that the corresponding array for 22×53 would have an extra row and that this would contain 53 dots, while the array for 21×54 would have an extra column and that this would contain 21 dots.

However, rather going straight to the array, some teachers invited students to come up with their own

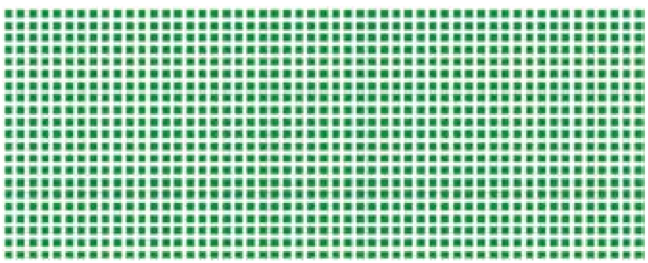

Figure 5: An array for 21×53.

Figure 6: Drawings of 21×53 and 21×54.

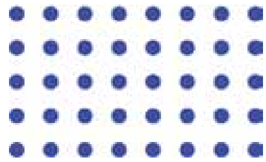

Figure 7: A 5 by 8 array.

drawings. This turned out to be both informative and productive. The students' drawings were usually structured more loosely than the array. However, the representations made sense to the students and were usually structured well enough to be fit for purpose. For example, in Figure 6 we have 21 circles (nicely arranged in rows of 10, 10 and 1) with '53' written inside each circle. We do not know what the 53 represents, perhaps it is biscuits, but perhaps for this student it does not really matter, they might be happy to work in a semi-abstract way. We then have the same arrangement of circles, but this time with '54' written in each, and '–1' by each circle to denote that the number in each circle has increased by 1, and so by 21 in all.

Although such informal models can help students to understand multiplication, we consider the array to be a much more powerful model. For example, by partitioning the array into two sub-arrays, it can model the distributive law that underlies long multiplication. Also, as it leads students towards the area model of multiplication, it can support students' understanding of multiplication (and division) of rational numbers. It is thus probably worth spending a lot of time, in primary and secondary school, working with the array. However, if we are to do this, it has to be done thoroughly. We tend to pay lip service to the power of the array but to underestimate what is required for students to discern its structure.

We have gradually come to the conclusion that the uniform appearance of the array, belies its complexity as a model of multiplication. The structure might seem obvious to us, but this is not necessarily so for students. Consider the array in Figure 7 and imagine a student who can see that the array has the same number of elements in each row and the same number in each column. Moreover, assume that the student sees that there are 8 elements in the top row and 5 in the the left hand column and so calls it a '5 by 8' array (or '8 by 5' perhaps). The student might still not see that this provides a model of 5 × 8. At present, the 5 and 8 are acting in similar ways, as 'composite units' (e.g. Steffe, 1992) that measure the number of dots in a single (or in each) row and column. However, if the array is to be seen as a model of 5 × 8, interpreted as '5 lots of 8', say, then just as in the earlier stories, one of the numbers, let's say the 5, has to take on a different meaning, as not the number of dots in each column but as the number of rows in the array – a 'composite, composite unit'.

There are some interesting studies with young children that identify some of the complexities of the array. For example, Outhred and Mitchelmore (2004) presented 6 – 9 year old students with a drawing of a 5 cm by 8 cm rectangle, with 1 cm marks drawn along the edges, and asked them to draw how the rectangle could be covered by 1 cm squares. They classified students' drawings into five 'spatial levels'. Examples of drawings from the first

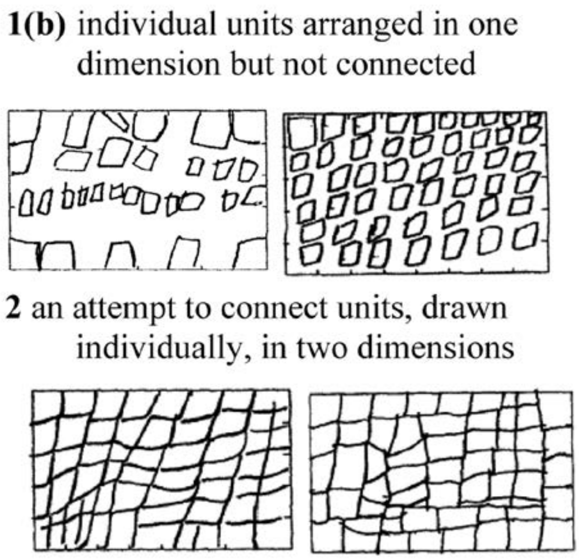

Figure 8: Students' drawings of a 5 by 8 rectangle covered in unit squares (from Outhred & Mitchelmore, 2004).

two levels are shown in Figure 8. As can be seen, the idea of structuring the rectangle into nice, neat rows and columns is not something that always comes easily and rapidly to young students.

We have observed something similar in secondary school. For example, Figure 9 shows a Year 7 student's attempt to represent how jelly babies could be arranged to cover a sheet of paper. The drawing suggests that this student has some sense of an array structure in that the 'jelly beans' have been drawn in columns. The numbers underneath the drawing suggest that the student sees the drawing as representing a '12 by 11' array, or at least something like it – there are 12 columns in the drawing, though some of the columns contain more than 11 dots. The calculation looks like an attempt at long multiplication, but without the step 'put down a 0' (or the understanding that multiplying by 11 can be performed by multiplying by 1 and 10 rather than by 1 and 1). What is striking here is that the student seems content to accept her calculated result (24), even though it bears no relation to the number of dots shown in her quite well-structured drawing. This suggests that the student does not yet realise how a drawing of this sort could help her test and make sense of her arithmetical procedures.

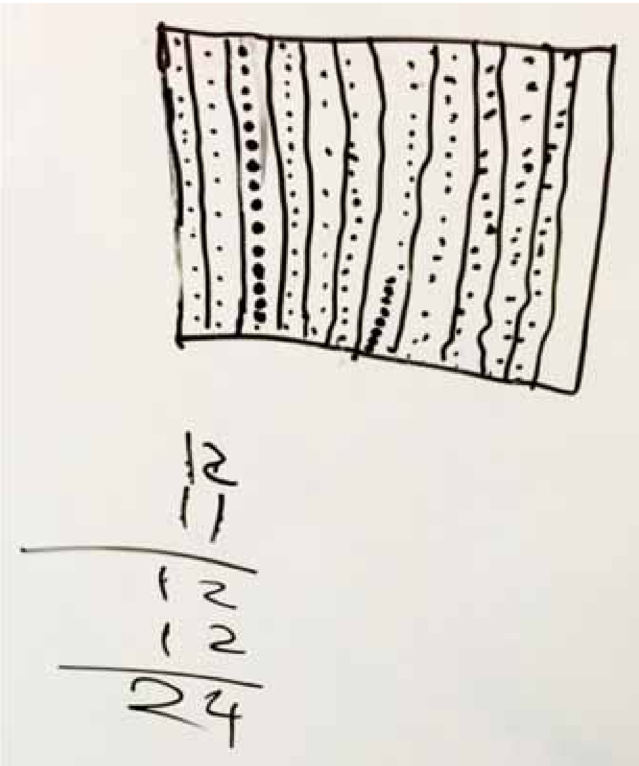

Figure 9: A drawing of a sheet of paper covered in jelly babies, and a calculation to find the total.

Conclusion

Our experiences of interviewing and of trialling lessons with Year 7 and 8 students, has confirmed to us that, facilitated by the teacher, models can play a vital role in giving meaning to mathematical concepts. This is perhaps recognised more clearly in primary school than in secondary school, where there is a tendency to downplay the role of some models - secondary students tend to think that the ones they met in primary school are 'babyish' and that they have grown out of them. However, we also need to realise, perhaps as much in primary as in secondary school, that models do not always give up their powers easily! Students need to spend considerable time with some models to fully see their structure and to be able to make productive use of them.

References

Anghileri, J. (1989). An investigation of young children's understanding of multiplication. *Educational Studies in Mathematics*, 20, 367–385.

Brown, M. & Küchemann, D. E. (1976). Is it an 'add' Miss?, Part 1. *Mathematics in School*, 5, 5, 15–17, Nov 76.

Outhred, L., & Mitchelmore, M. (2004). Students' structuring of rectangular arrays. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th PME* International Conference, 3, 465–472.

Steffe, L. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences, 4(3)*, 259–309.

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** In the Cartesian product interpretation of multiplication, the numbers can both refer to apples. For example, "There are 9 red apples and 3 green apples. I want a red and a green apple. How many pairs can I choose from?".

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