# Mathematical argument, language and proof 

## AS/A LEVEL 2017

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The reformed Mathematics AS/A level Content (ref.1) for first teaching in 2017 was published by the Department for Education (England) in April 2016. The Content states that reformed AS/A level specifications in Mathematics must require students to demonstrate a wide range of overarching knowledge and skills, and that these must be applied, along with associated mathematical thinking and understanding, across the whole of the detailed Content, and that the three Overarching Themes:

- OT1 Mathematical argument, language and proof
- OT2 Mathematical problem solving
- OT3 Mathematical modelling
must permeate the Content.
In respect of the first Overarching Theme, three specific aims and objectives required of the reformed qualifications are that AS/A level specifications in Mathematics must encourage students to:
- reason logically and recognize incorrect reasoning;
- generalize mathematically;
- construct mathematical proofs.

The Content pertinent to these objectives under OT1 is set out below (in bold) and applies to both AS and A level (except OT1.4, which is not in bold and applies to A level only):

Importantly, the intention of the reforms is most definitely not to equate OT1 with 'proof' or to treat 'proof' as a separate teaching topic. The emphasis is intended to be on students being able to know and use correct mathematical language and argument across the whole of the Content. Of course one way to encourage this is through engagement with formal proof. As such, it is important that students are exposed to a range of examples, particularly ones requiring little more than higher level GCSE, to help with their development of knowledge, skills and understanding associated with OT1. This will enable students to make progress in demonstrating the overarching knowledge and skills, which are then to be applied, along with associated mathematical thinking and understanding, across the whole of the detailed Content as stated in ref. 1, for example to calculus, trigonometry, and so on.

This article provides a range of examples involving different types of proof and mathematical argument, with the aim of supporting students taking their first steps into this vital overarching theme in mathematics. Aside from learning about different techniques of proof, and developing problem solving approaches, the main purpose of the reforms is to encourage students to develop logical thought, and to be able to provide clear mathematical arguments in support of a result, using correct mathematical notation and language. Again, this is the primary purpose of OT1.

Before considering some examples, we note that the relevant Detailed content statements in ref. 1 include reference to 'proof', as shown below, in which there is explicit reference to different types of proof.

We see the 'classic' examples cited in the Detailed content: proof of the 'irrationality of $\sqrt{2}$ ' and 'infinity of primes', but then 'application to unfamiliar proofs'. While it is this last phrase that will concern teachers and students, hopefully providing some ideas of examples of what might constitute 'unfamiliar proof' might allay some of the concerns.

## 0T1 Mathematical argument, language and proof

|  | Knowledge/Skill |
| :--- | :--- |
| OT1.1 | Construct and present mathematical arguments through appropriate use of diagrams; sketching <br> graphs; logical deduction; precise statements involving correct use of symbols and connecting <br> language, including: constant, coefficient, expression, equation, function, identity, index, term, <br> variable. |
| OT1.2 | Understand and use mathematical language and syntax as set out in the content. |
| OT1.3 | Understand and use language and symbols associated with set theory, as set out in the content. <br> Apply to solutions of inequalities and probability. |
| OT1.4 | Understand and use the definition of a function; domain and range of functions. |
| OT1.5 | Comprehend and critique mathematical arguments, proofs and justifications of methods and <br> formulae, including those relating to applications of mathematics. |

## A Proof

|  | Content |
| :--- | :--- |
| A1 | Understand and use the structure of mathematical proof, proceeding from given assumptions <br> through a series of logical steps to a conclusion; use methods of proof, including proof by deduction, <br> proof by exhaustion. |
| Disproof by counter-example. <br> Proof by contradiction (including proof of the irrationality of $\sqrt{2}$ and the infinity of primes, and <br> application to unfamiliar proofs).$\quad .$and |  |

Before we give examples for readers to use with their students, possibly as a focus for discussion/exploration, we include examples of each of four 'methods of proof/ disproof' as given in the Content statement above.

For some of the examples the following definitions are required:
The set of natural numbers is denoted by $\mathbb{N}=\{1,2,3, \ldots\}$.
The set of integers is denoted by $\mathbb{Z}=\{0, \pm 1, \pm 2, \pm 3, \ldots\}$.
The set of positive integers is denoted by $\mathbb{Z}^{+}=\{1,2,3, \ldots\}$. The set of rational numbers is denoted by

$$
\mathbb{Q}=\left\{\frac{p}{q}: p \in \mathbb{Z}, q \in \mathbb{Z}^{+}\right\} .
$$

The set of real numbers is denoted by $\mathbb{R}$ and can be thought of as all numbers on a number line.
A real number that is not a rational number is called an irrational number.

A prime number is a positive integer $p>1$ that has no positive integer factors other than 1 and $p$ itself.
A square number is a number of the form $n^{2}$ where $n \in \mathbb{Z}$. A cube number is a number of the form $n^{3}$ where $n \in \mathbb{Z}$.

## 1. Example of proof by deduction

Prove that for all $n \in \mathbb{Z}$, if $n$ is odd, then $n^{2}$ is odd.
Proof
Let $n$ be an odd integer, then $n=2 k+1$ for some $k \in \mathbb{Z}$.
Therefore $n^{2}=(2 k+1)^{2}=4 k^{2}+4 k+1=2\left(2 k^{2}+2 k\right)+1$.
Now, since $k \in \mathbb{Z}$ we can write $n^{2}=2 m+1$, where $m=2 k^{2}+2 k \in \mathbb{Z}$, and hence $n^{2}$ is odd, as required.

## 2. Example of proof by exhaustion

Prove that every integer that is a perfect cube is either a multiple of 9 , or 1 less, or 1 more than a multiple of 9 .

## Proof

Each cube number is the cube of some integer $n$. Every integer is either a multiple of 3 or is one less or two less than a multiple of 3 since the maximum remainder when
dividing by 3 is 2 . The set of integers can therefore be divided into three non-overlapping cases which are exhaustive.

Case 1: If $n=3 m$ for some $m \in \mathbb{Z}$, then $n^{3}=(3 m)^{3}=27 m^{3}$ $=9\left(3 \mathrm{~m}^{3}\right)$, which is a multiple of 9 .

Case 2: If $n=3 m-1$ for some $m \in \mathbb{Z}$, then $n^{3}=(3 m-1)^{3}=$ $27 m^{3}-27 m^{2}+9 m-1=9\left(3 m^{3}-3 m^{2}+m\right)-1$, which is 1 less than a multiple of 9 .

Case 3: If $n=3 m-2$ for some $m \in \mathbb{Z}$, then $n^{3}=(3 m-2)^{3}=$ $27 m^{3}-54 m^{2}+36 m-8=9\left(3 m^{3}-6 m^{2}+4 m-1\right)+1$, which is 1 more than a multiple of 9 .

Therefore the result is true for all integers $n$, as required.

## 3. Example of disproof by counter-example

Disprove the statement that: for all $n \in \mathbb{Z}$, the integer $f(n)=n^{2}-n+11$ is prime.

Proof
We disprove this statement by finding an example for which the statement does not hold, which is called a counter-example.

While $f(n)$ is prime for some values of $n$ as shown in the Table, when $n=11, f(11)=11^{2}-11+11=11 \times 11$, and hence not prime, so that the statement is not true as there exists an $n \in \mathbb{Z}$, namely $n=11$, for which $f(n)$ is not prime.

This provides a counter-example to the statement, and hence we have disproved the statement, as required.

## 4. Example of proof by contradiction

Prove that for all $n \in \mathbb{N}$, if $4^{n}-1$ is prime, then $n$ is odd.
Proof
We prove this by contradiction, as follows.
Suppose that this is not the case, i.e. that $4^{n}-1$ is prime, where $n \in \mathbb{N}$, and $n$ is even.

| $n$ | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $f(n)$ | 23 | 17 | 13 | 11 | 11 | 13 | 17 | 23 | 31 | 41 | 53 | 67 | 83 | 101 |

In this case, $n=2 k$ for some $k \in \mathbb{N}$, and hence $4^{n}-1=4^{2 k}-1$ $=\left(4^{k}\right)^{2}-1=\left(4^{k}-1\right)\left(4^{k}+1\right)$.

However, since $k \in \mathbb{N}$, we have that both $4^{k}-1,4^{k}+1 \in \mathbb{N}$ and $4^{k}-1,4^{k}+1>1$, and hence $4^{n}-1$ has at least two factors that are greater than 1 , which contradicts the fact that $4^{n}-1$ is prime.

Therefore our assumption that $n$ is even is incorrect, and hence $n$ must be odd, as required.

## Examples

Before starting these, recall Polya's four-point plan for solving any type of problem:
(i) understand the problem;
(ii) devise a plan;
(iii) execute the plan; and
(iv) look back.

## In each case either prove or disprove (by finding a counter-example) the given statement.

1. For all $n \in \mathbb{N}$, if $n$ is odd, then $n^{2}$ is odd.
2. For all $n \in \mathbb{N}$, if $n^{2}$ is odd, then $n$ is odd.
3. For all $n \in \mathbb{N}, n$ is the sum of squares of two natural numbers.
4. The sum of two odd numbers is even.
5. The sum of two even numbers is even.
6. The sum of an even number and an odd number is odd.
7. The product of two odd numbers is odd.
8. For all $n \in \mathbb{N}$, if $n$ is even, then $9 n^{2}+6 n$ is a multiple of 12 .
9. The sum of two consecutive odd numbers is equal to the difference of two square numbers.
10. Any natural number greater than 1 (so $2,3,4, \ldots$ ) has a prime factor.
11. The product of two consecutive integers is even.
12. There are infinitely many prime numbers.
13. For all positive real numbers $a, b, \mathrm{c}$ such that $a b=c$, then $a \leq \sqrt{ } c$ or $b \leq \sqrt{ } c$.
14. For all $n \in \mathbb{Z}$, the integer $f(n)=n^{2}-n+11$ is prime.
15. For all $m, n \in \mathbb{Z}$, if $m^{2}+n^{2}$ is even, then $m+n$ is even.
16. For all $n \in \mathbb{N}$, the sum of $n$ consecutive natural numbers is:
(a) even if $n$ is a multiple of 4 ;
(b) odd if $n$ is even but not a multiple of 4;
(c) divisible by $n$ if $n$ is odd.
17. If $x$ is a real number such that $\underline{x^{2}-1}>0, x \neq-2$, then either $x>1$ or $-2<x<-1 . \quad \overline{x+2}$
18. No positive real number $a$ exists such that $a+\underline{1}<2$.
19. No integer $n$ exists such that $4 n+3$ is a square number.
20. For all non-negative real numbers $a, b, c$, if $a^{2}+b^{2}=c^{2}$, then $a+b \geq c$.
21. For all $n \in \mathbb{Z}, n(n+1)$ is even.
22. For all real numbers $x, y, 2^{x} 2^{y}=2^{x y}$.
23. For all $n \in \mathbb{Z}$, if $n$ is odd, then $n^{3}$ is odd.
24. For all $n \in \mathbb{N}$, if $4^{n}-1$ is prime, then $n$ is odd.
25. For all $m, n \in \mathbb{Z}$, if $m n$ is even, then either $m$ or $n$ is even.
26. For all $n \in \mathbb{N}$, if $2^{n}-1$ is prime, then $n$ is prime.
27. For all $a \in \mathbb{Q}$ and all irrational numbers $b, a+b$ is irrational.
28. If $a+b>100$, then either $a>50$ or $b>50$.
29. For all $a \in \mathbb{Q}$, and all irrational numbers $b$, then $a b$ is irrational.
30. There is no largest even integer.
31. $\sqrt{2}+\sqrt{6}<\sqrt{15}$.
32. $\sqrt{6}-\sqrt{2}>1$.
33. For all $n \in \mathbb{N}$, if $n^{2}$ is even, then $n$ is even.
34. For all $n \in \mathbb{N}$, if $n^{2}$ is a multiple of 3 , then $n$ is a multiple of 3.
35. For all $n \in \mathbb{N}$, if $n=m^{3}-m$ for some $m \in \mathbb{N}$, then $n$ is a multiple of 6 .
36. If $n, k$ are positive integers, then $n^{k}-n$ is always divisible by $k$.
37. Between any two rational numbers there is another rational number.
38. There exists a real number $a$ such that $a^{2}=2$. (Hint: Draw a right-angled triangle.)
39. If $n \in \mathbb{N}$ there exists a real number $a$ such that $a^{2}=n$. (Hint: Draw a circle of diameter $n+1$ and the perpendicular through a point on the diameter which divides the diameter in the ratio $n: 1$.)
40. $\sqrt{2}$ is irrational.
41. There exists irrational numbers $a, b$ such that $a^{b}$ is rational. (Hint: Consider $\sqrt{2}^{\sqrt{2}}$.)
42. Between any two real numbers there is an irrational number.
43. $\sqrt{3}$ is irrational.
44. The product of two rational numbers is always rational.
45. The product of two irrational numbers is always irrational.
46. For all real numbers $u, v$, if $0<u, v<1$, then $\frac{u+v}{1+u v}<1$.
47. $\sqrt{ } 4 \in \mathbb{Q}$ is irrational. (Note: $\sqrt{4}=2=\frac{2}{1}$ and so is clearly rational! Your challenge is to attempt to prove the original statement by trying to use the same proof by contradiction argument that you would for proving that $\sqrt{ } 2$ is irrational and identify where the argument fails in this case as you hope it would do.)
48. Every integer that is a perfect cube is either a multiple of 9 , or 1 less, or 1 more than a multiple of 9 .
49. For all real numbers $a, b, c$, if $a+b+c=0$, then $a b+b c+c a \leq 0$.
50. For all non-negative real numbers $a, b$, then $\frac{1}{2}(a+b) \geq \sqrt{a b}$.
51. For all $n \in \mathbb{Z}, n^{2}+3 n+7$ is odd.
52. For all $p \in \mathbb{Q}, p \neq 0$, and all irrationals $q, p q$ is irrational.
53. For all $n \in \mathbb{N}, n^{2}-n$ is a multiple of 3 .
54. The square of any natural number is of the form $3 k$ or $3 k+1$ for some $k \in \mathbb{N}$.
55. There are no natural numbers $m, n$ for which $m^{2}-n^{2}=1$.

This list is available via the link: www.personal.reading. ac.uk/~smsglais/Examples-of-proof.pdf. Readers are invited to contact the authors by email at p.glaister@ reading.ac.uk with further examples and these will then be added to the list.

Finally, readers may be interested to note that 2017 is a prime number, and that $2017=9^{2}+44^{2}$, the sum of squares of two natural numbers. In fact, any prime number of the form $4 n+1$, where $n \in \mathbb{N}$, can always be expressed uniquely as the sum of squares of two natural numbers. More generally, there is a theorem, originally written down by Fermat and subsequently proved by Euler (ref.2), which states that all odd prime numbers $p$ (i.e. not including $p=2$ ) can be expressed uniquely as the sum of squares of two natural numbers $x, y \in \mathbb{N}$, i.e. $p=x^{2}+y^{2}$, if and only if $p=4 n+1$ for some $n \in \mathbb{N}$.

The last year for which this occurred was 1997 ( $=29^{2}+34^{2}$ ), and this was also true for the previous 'prime year', $1993\left(=12^{2}+43^{2}\right)$. Readers may wish to determine the next 'prime year' for which this is true, as well as the next time two consecutive 'prime years' have this property. All of these are in this century, but are still some way off!

Similarly, when was the last time, and when is the next time that three 'prime years' in a row have this property? The last time the latter occurred was just after Fermat died, and the next time the latter occurs there are three successive occurrences of this. After this, the next time 'prime years' in a row have this property there are five in a row. For four 'prime years' in a row with this property we have to go back to the time of the Magna Carta!

Readers are advised not to add the last theorem to the list of examples above as the proof is certainly beyond A level!

## References

1. www.gov.uk/government/publications/gce-as-and-a-levelmathematics. Accessed January 12017.
2. mathworld.wolfram.com/Fermats4nPlus1Theorem.html. Accessed January 12017.

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