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Learning from others
Can PISA and TIMSS really inform curriculum developments in mathematics?

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One of the problems that will vex any President of the Mathematical Association is the topic of the address with which he or she closes his or her year of office. This occupied me, on and off, for more than a year. In my case, in addition to my desire to acknowledge the honour of the invitation made to me, I was deeply conscious of the fact that I would be the 100th individual to serve as President. I dabbled with some pet themes, typically concerning the lack of genuine problem-solving or proof in English school mathematics, before concluding that the most sensible thing would be to talk on the topic about which I know most. My research interests are in comparative mathematics education. I have been fortunate, over the last twenty years or so, to have been able to visit and videotape mathematics classrooms in several European countries. In so doing I have had my understanding of mathematics teaching transformed in ways that led, almost inevitably, to the theme of both this talk and the conference which brought my Presidency to an end: *Learning from Others*.

My first visits to mathematics classrooms outside the UK occurred in the early 1990s. As a consequence of a European Union student exchange scheme, the late Gill Hatch and I made several visits to schools in Budapest. Arranged by colleagues at our partner institution, Eötvös Loránd Tudományegyetem or ELTE, we observed lessons in various schools throughout the city. At that time Gill and I were deeply immersed in Cockcroftian perspectives on classroom practice and, in typical English Imperialist manner, assumed we would have little to learn and everything to offer. Such arrogance, such naivety! We saw lessons of such mathematical sophistication and didactical elegance, even with students in vocational schools, that our parochial perspectives on mathematics teaching and learning were so completely transformed that we would never be the same again. It was a professionally Damascene moment and 'learning from others' remains a guiding principle in all aspects of my life.

As I write this piece, the English are in the middle of curriculum reform. This tends to happen each time we experience a change of government and is something that amuses, in a black humour sort of way, my European colleagues. They ask: does each new government really believe that systemic change will allow schools to function adequately? Why is curriculum content not immune from the whims of politicians? Why do the English authorities appear only to listen to users of mathematics? Surely the curriculum should be developed by those who know and understand the subject and not politicians, industrialists, economists, engineers and scientists who collectively can create only anarchy. In this respect, one has only to examine, for example, the composition of the steering group for Adrian Smith's post-14 mathematics inquiry to see the extent of the problem: users of mathematics abound and mathematicians themselves are hard to spot.* Such questions remind one that colleagues in other countries frequently operate in more stable and less politically controlled contexts and may, as a consequence, be able to induct their students more successfully into the ways of mathematics.

Of course, if everything were rosy in the garden of English mathematics then there would be a clear warrant for politicians remaining outside curricular content debates. Unfortunately, everything is not rosy, although successive governments seem reluctant to admit that faults may lie in their constant appeasement of incompatible and powerful interest groups. English primary mathematics has been demeaned by the label 'numeracy'. In secondary schools, due to repeated systemic failures to embed proof, mathematics is frequently construed as cognate to science, which clearly it is not. However, my concerns are not unique and some outsiders have offered some rather telling insights. One study [1] characterised English classrooms in which students worked individually on booklets or work-cards in the following manner. Lessons began with teachers completing various administrative tasks including the register. They then circulated the room to help students with difficulties and record their latest test marks. Finally, a few minutes before the bell, they would invite students to pack away. In response, one feels compelled to ask, where was the mathematics in his description and, more worryingly, where was the mathematics in the experiences of the students he observed?

My experiences of such classrooms in my early days of teacher education yielded the following incident. A trainee teacher, increasingly frustrated by the queues of students forming at her desk for help, bought a book of cloakroom tickets. In the manner of a supermarket delicatessen, she informed her students that if they got stuck they were to collect a ticket, return to their seats and, while waiting their turn, try to solve their problems for themselves. On one occasion, a boy took a ticket, returned to his seat and, seeing a solution to his problem, realised he no longer needed his place in the queue. He waved his ticket in the air and asked, animatedly, does anyone want number thirty-seven for ten pence? Sadly, he got two takers, who then argued over whose need was greatest.

More recently, others [2, 3] have described English mathematics as 'pragmatic'. New concepts or methods are typically given as information or in the style of a recipe. Theorems are warranted by experiment or teacher assertion rather than proof. Precise language is seen as unimportant and standard algorithms are subordinated to students' own methods. Again, in response, one asks, how can learners gain ownership over concepts if they are presented as recipes. Where is the systematic encouragement of their engagement in the processes of conceptual development? How, in any mathematical world I know, can theorems be warranted by experiment or

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* See http://www.mathsinquiry.org.uk/steering.html#SS

assertion? A theorem without proof remains a conjecture. Mathematics without proof *is* science. Is it any wonder that so many English schools conflate the two subjects?

Such problems are compounded by the text books students typically use. Comparisons of English, French and German textbooks [4, 5] found that the English comprise shorter chapters with fewer words and examples per page. They incorporate little explanatory text, although there may be worked examples to overcome the need for teacher mediation. They place little emphasis on technical vocabulary, and comprise exercises, typically of questions similar to the worked example, with little cognitive demand or obvious scope for extension. If, and the evidence suggests this is the case, English textbooks are so poor in relation to those produced in similar cultural contexts, then perhaps it is time for government to accept appropriate regulation.

Teachers' beliefs about mathematics, its curricular importance and teaching present an equally dismal picture. An interview survey [6, 7] of teachers from different parts of England found two-thirds of respondents espousing a belief that the curricular importance of mathematics lies in providing students with the skills of applicable number preparatory for a world beyond school. Obviously, there were other perspectives. For example, a fifth of the cohort expressed beliefs commensurate with a view of mathematics as an exploratory and problem-solving discipline; a third of the cohort believed that the curriculum was a given and that either they should not be expected to have opinions as to its content or that pursuing topics of particular interest to them may prejudice their students' opportunities; a quarter saw the mathematics curriculum as a function of the children they taught – the more able the child the more he or she should experience mathematics rather than, essentially, numeracy. By way of contrast, the same study found the whole Hungarian cohort arguing that students should learn about the structural properties of mathematics and, importantly, how to think mathematically and to solve problems. Even when they alluded to preparing students for a world beyond school they spoke about the skills of logical thinking derived from learning mathematics. This very different Hungarian perspective on mathematics presents me with an opportunity to introduce into the discussion a problem I encountered on one of my first visits to Budapest.

A grade 6 teacher began his lesson, in typical Hungarian fashion, with a public discussion of a problem posed the previous day. It was based on determining how many isosceles triangles of area 9 cm^2, with each vertex located at a Cartesian grid point and one, in particular, at (3, 1) could be found. Collectively, with the teacher doing little but manage students' contributions, the class collectively agreed that the key dimensions of the triangle, base and height, had to be factor pairs of eighteen; the base had to be even in order for the third vertex to lie on a grid point; and that three families – base 2 and height 9, base 6 and height 3, base 18 and height 1 – satisfied these criteria. Finally, for each family, the point (3, 1) could be any of the three vertices and, having fixed a particular vertex, rotations about that vertex would yield a further three triangles to give four triangles for each vertex at (3, 1). Thus, each family comprised twelve triangles, giving a total of 36. Such problems exemplify a mathematics education tradition with problem-solving at its heart. Such problems integrate several topics in ways that allow for frequent revision of ideas covered earlier; they offer learners opportunities to engage in high level mathematical reasoning and represent a very different perspective on mathematical exercises from those described above by Kaiser and her colleagues.

Thus, from various perspectives, English children appear to experience an intellectually impoverished form of mathematics. A recent video study of mathematics teaching in England, Flanders, Hungary and Spain found that, in comparison with their European counterparts, English children rarely experienced mathematics as a connected body of knowledge or notions of efficiency and elegance as worthwhile objectives [8]. Moreover, English teachers, while exploiting some didactical strategies like explaining and coaching in proportions comparable to their European colleagues, rarely ask questions that challenge students' thinking or exploit students' prior knowledge in ways that help them see connections within and between topics [9]. Thus, if English mathematics is in crisis, and I believe it is, it would seem appropriate to look to other systems for possible solutions. However, knowing where to look is another problem altogether.

In this respect it was encouraging to note, as part of its approach to curriculum reform, that the current government has indicated explicitly an interest in how ‘high performing jurisdictions’ present mathematics to their children. One obvious temptation is to examine the results of TIMSS* and PISA† for likely candidates and the government has, for reasons best known to itself, identified Singapore as one source of ‘adaptive potential’ [10]. However, I argue that Singapore is so culturally different from the UK as to make it a worthless location for understanding how systemic mathematics success can be achieved. Singapore is unique. It is small. At 697‡ km^2, it is

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* TIMSS, the Trends in International Mathematics and Science Study, is a large scale international test of mathematics and science achievement first undertaken by the International Association for the Evaluation of Educational Achievement in 1995 and repeated every four years. The outcomes of TIMSS 2011 are due the end of 2012. In respect of mathematics, its primary focus is the assessment of students' abilities to perform standard mathematical tasks in grades 4 (Year 5) and 8 (Year 9). In my opinion, TIMSS is able to alert us well to how our students perform of different topics.

† PISA, the Programme of International Student Assessment, is a large scale international assessment of 15-year old student literacy, mathematical literacy and scientific literacy. Managed by the Organisation for Economic Cooperation and Development, it was first undertaken in 2000 and is repeated every three years. It is an attempt to assess students' subject competences in authentic or real-world contexts. PISA is increasingly being used as a benchmark of a system's success. In my opinion, due to the nature of its assessment items, an over-reliance on the results of PISA to drive policy is misguided.

‡ All figures can be found at https://www.cia.gov/library/publications/the-world-factbook/index.html

one of the smallest countries in the world and yet, at $59,900, it has the fifth highest per capita GDP in the world. By contrast, at $35,900, the UK's per capita GDP ranks 34th on the international scale. We are poor by comparison. Moreover, when its neighbours, Malaysia ($15,600), Thailand ($9,700) and Indonesia ($4,700), are considered, it is not difficult to see that there is something special about Singapore. And one of the most special things about Singapore is that

> "low-paying… jobs are done by Malaysians, who daily cross the two bridges connecting the countries to work in Singapore; by large numbers of Philippine women who come to work in Singapore, leaving their families behind… who, even in low-paying Singapore jobs, can earn far more than they can in their home countries. These workers have children, but their children are tested in their home countries, not in Singapore." [11, p. 668].

In short, a good curriculum incorporating, for example, opportunities for problem-solving at all levels of the student experience, which Singapore appears to have [12], is not necessarily a guarantee of systemic success. Being rich may help, although this is not always the case, as an examination of Norway's performance on international tests shows.

A country typically construed as worthy of investigation is Finland. So well publicised have been its successes on the four reported iterations of PISA that it has attracted envoys from around the world [13]. Among these has been Ofsted, England's schools' inspectorate, whose visits have culminated in reports on the teaching of mathematics [14, 15], indicating, in the oblique manner of civil servants, that repeated Finnish success may not be due to pedagogical excellence and the quality of the mathematical challenges presented to students. Interestingly, the Finns themselves have attributed their PISA-related successes to various factors. Firstly, they have lauded, with justification, the comprehensive school system, which is based on equity for all, irrespective of gender, social status or ethnicity, and a compulsory nine year basic curriculum [16, 17]. Providing education to age 16, a typical comprehensive school is local to the student, small, well-equipped [18] and sufficiently funded for it to provide free school meals for all [13]. Students, who are neither tracked [19, 20] nor streamed [21, 18], are taught in schools typically construed as learning and caring communities [22, 23]. Interestingly, the right to choose the school their children attend has had little influence on parents' decision-making [24] and reflects what can only be construed as an enviable provision in a country in which there is, effectively, no independent educational provision.

A second factor concerns the extensive and integrated provision for special educational needs students. This involves around 1 in 6 students, requires no formal statement of need, begins when difficulties arise and has both reduced the stigma of special needs and promoted inclusion [21, 25]. It is typically focused on supporting pupils' mother tongue and basic mathematical skills [26, 27]. Importantly, since PISA and Finnish special education both focus on the same areas – literacy and mathematics problems located in a textual narrative − "it seems plausible that the special educational system in this country plays a positive role in relation to PISA" [26, p. 276], not least because Finnish students' mathematical word problem competence is a function of their reading competence [28].

Thirdly, Finnish teachers are talented and continue to enjoy high public esteem [16, 23, 29, 30], being considered professionals who know what is best for their students [22]. Teaching is a popular career choice among school leavers, even though fewer than 20% of applicants are successful [13, 31]. A master's degree, requiring 4 to 5 years to complete, is an essential prerequisite [13, 19, 31, 32], ensuring "an academically high standard of education for prospective teachers" [33, p. 29], a condition that has been well received by teachers who see it as enhancing their professional status [32].

Thus, it would seem that the Finns have created an educational infrastructure appropriate for high levels of systemic achievement, which has been matched not only by high mean PISA scores but also the smallest between school variation of all OECD countries [20, 21, 34, 35]. However, despite such, rather heroic, claims, there are dissenting voices among the academic mathematics communities concerned that PISA success masks structural problems. For example, Finland has participated in one of the four TIMSS whose results have been published. TIMSS 1999, as have all the TIMSS studies, assessed students' technical competence on five broad topic areas of a typical mathematics curriculum. While Finnish overall grade 8 performance, 520, was average by European standards its performance on geometry, 494, and algebra, 498, was poor and construed by Finnish academics as reflecting a decline in the mathematical knowledge necessary for students to continue to higher education [36, 37], not least because earlier deductive approaches to mathematics have been replaced by procedural approaches that have marginalised logical thinking, elegance, structure and proof [38].

So, what sense can be made of this apparently conflicting evidence? One answer lies in the nature of PISA's assessment processes. PISA assesses 15 year-old students' application of subject knowledge – mother tongue, mathematics and science − to authentic or real-world situations [39]. Its problems, typically incorporating elementary mathematical topics, are set in textual narratives. The evidence indicates that Finnish students are better able to extract from the text and perform accurately these simple mathematical processes than students elsewhere. However, this typical form of PISA test item implies that success may be more a function of a student's comprehension than mathematical expertise. Not only would this explain Finnish PISA success but also why its performance on TIMSS was, in relative terms, poor. Support for this conjecture can be found in an analysis of Finnish culture. The Finnish people have a strong collective identity deriving from successive periods of Swedish and Russian colonialism lasting from the mid-thirteenth century until independence in 1917 [33]. For more than four hundred years reading competence was a prerequisite for receiving Lutheran sacraments; failure in the public examination, or *kinkerit*, meant a denial of permission to marry. Consequently, Finland has, essentially, no illiterate underclass and takes pride, collectively, in its being a culture with an appreciation for education and high expectations with respect to personal responsibility [21, 40] − the Finnish library network is among the densest in the world, with Finns borrowing more books than anyone else [23].

This deep-seated cultural influence is highlighted further when one looks beyond headline figures Finnish PISA performance. Finland is largely monocultural although, in addition to the traditionally itinerant Sami population in the north, there is a substantial Swedish-speaking minority in the west. The Swedish-speaking community is a legacy of colonial times and is a particularly powerful economic group. For example, despite accounting for less than six percent of the population, Swedish-speaking Finns account for a disproportionate 24 percent of board members of the 50 largest companies listed on the Helsinki stock exchange [41] and invest, per capita, three times as much in shares as Finnish-speaking Finns [42]. In such circumstances, as would be the case almost anywhere else in the world, it would be reasonable to expect such an economically elite group to perform more highly on tests of educational achievement than the rest of the community. Yet, the figures of table 1 show clearly that this is not the case.

Finnish-speaking Finns	Swedish-speaking Finns	Swedish-speaking Swedes
541	527	494

TABLE 1: PISA 2009 mathematics literacy scores for three cultural groups

Despite being taught the same curriculum in equally well-resourced Swedish language schools [43], Swedish-speaking Finns perform significantly poorer on PISA and internal Finnish assessments than Finnish-speaking Finns [44] − a finding compounded by evidence that Swedish-speaking Swedes perform even more poorly. Thus, one concludes that there must be something about the Finnish-speaking community that not only distinguishes it from others, not least because in most countries the economically powerful typically achieve more highly than the rest of the population but also makes it a less than helpful site for an exploration of adaptive potential.

Additionally, when analyses of Finnish classrooms are taken into consideration, there is little evidence to suggest that Finnish teachers, despite the quality of their schools and professional training, are pedagogically exceptional. For example, Finnish teaching has been described as a teacher-dominated practice that has changed little in fifty years [45]. Despite various curricular reforms advocating a “diversification of teaching methods” alongside a shift of emphasis from “routine skills onto development of thinking” [46, p.11], an external review commissioned by government found a conservative workforce, uncertain how to adapt to expected changes, continuing to teach as it always had [47]. More recently a prime ministerial initiative aimed at improving mathematics teaching through substantial in-service opportunities for teachers seems to have resulted in little change [48], particularly with respect to systemic expectations of mathematical problem-solving [49].

Such problems remain. For example, an interview study of Finnish teacher educators found a concern that many teachers, sticking too rigidly to the textbook, present 'boring mechanical' lessons [50]. Moreover, the same study found evidence that, despite initial enthusiasms for alternative approaches to mathematics teaching, many newly qualified teachers slide into similar ways of working. Such perspectives resonate with my own experiences of Finnish classrooms derived from analyses of videotaped sequences of lessons taught by competent teachers on topics typically taught to students in the age range 10-14. Finnish teachers appear to work within a tradition of implicit didactics. They appear to offer definitions and model solution strategies without ever making things explicit. They rarely, if ever, seek or offer clarification during public discourse, typically leaving students to infer whatever meaning they can from such exchanges. To support such inferences, teachers frequently encourage, implicitly and explicitly, students to make extensive notes, supported by their writing extensively on the board, typically very slowly in capital letters. Moreover, there was evidence, in several lessons, of an expectation that students would discuss their notes and sense-making at home [51, 52, 53].

By way of example, an introductory lesson on percentages played out in the following manner. Markku, the teacher, began by writing, very slowly and in silence, *Prosentti (=sadasosa)**, before commenting that *there are three ways to denote one hundredth. Write these three ways in your notebook; write them in an equation form*. After about a minute, Salla, having been invited to the board, wrote without hesitation but also slowly and deliberately, *1/100 = 1% = 0,01*. Markku thanked Salla before asking his students for examples of one hundredth from everyday life. Part of the ensuing conversation went as follows:

Markku	Where have you met one hundredth. Liisa?
Liisa	The circumference of the earth compared with the circumference of the sun
Markku	Yes… Simo?
Simo	One hundred percent fat.
Markku	Yes… But it can also be one percent fat. Leena?
Leena	I have not actually seen one hundredth, but I have seen percentages in election results.
Markku	Yes… For instance there are percentages in alcoholic drinks. They contain more than one percent.

In this exchange Markku offered no explicit definition but seemed, by

* Percent (=hundredth)

way of implicit definition, to draw on Salla's tripartite equation. His reactions to students' responses seemed ambiguous. For example, when thanking Salla for her equation it was not clear whether her response was accepted as accurate or not, for he neither offered nor sought further comment. His reaction, *yes*, to Liisa's and Leena's responses left an observer confused as to what he intended his students to take from the exchange as, in both cases, he followed his utterance with a change of direction unrelated to the contributions made. Indeed, the shift of attention from election percentages to alcoholic drinks seemed particularly strange, especially when viewed against the ages of his students. Moreover, having asked a question about his students' awareness of one hundredth in real life, he seemed content for this to become a discussion on percentages.

Thus, if Finnish mathematics education is less exemplary than PISA implies, are there other European systems worthy of study? In this respect, I have alluded already to the qualities of Hungarian mathematics education, but Hungarian performance on PISA is only moderate, although its TIMSS-related achievements have typically above the European average. However, if we are to assume that the results of international tests are indicators of possible sites of interest, then the figures of Table 2 may be of some help.

	PISA (Age15)				TIMSS (Age 14)			
	2000	2003	2006	2009	1995	1999	2003	2007
Finland	536	544	546	541	---	520	---	---
Flanders	543	553	541	537	565	558	537	---

TABLE 2: Finnish and Flemish mathematics performance on the published PISA and TIMSS tests.

From the perspective of PISA it can be seen that Flemish students performed as least as well as the much lauded Finnish. In fact, the mean Flemish score over the four cycles is higher than the Finnish. Moreover, on the three cycles of TIMSS on which it participated, Flanders was the highest performing European system. So why has Flanders not been subjected to the same scrutiny as Finland? I think there may be two reasons. Firstly, Flemish performance on PISA is not widely known because the OECD reports Belgium as a whole, a process which has masked Flemish performance. For example, the Belgian scores over the four PISA cycles were 520, 529, 520 and 515 respectively. Fortunately, academics at the University of Ghent have produced detailed reports on Flemish performance after each PISA cycle [54, 55, 56, 57] and it is from these that the Flemish data have been extracted. Secondly, the Flemish have not engaged in the self-promotion activities of the Finns. For example, a news briefing from the Finnish London embassy announced that "Finland is the most successful nation in the world in educating children" (January 14, 2008), which reflects the tone adopted in much of the Finnish research published over the last decade – see some of the titles of some of the Finnish articles in the reference list below. Nothing in such a triumphant mode has emerged from Flanders.

So, what is known about Flanders and its mathematics achievements? Firstly, despite its relatively small population, it is densely populated and more obviously multicultural than Finland. While nominally comprehensive, Flanders operates a school system differentiated according to whether schools are explicitly secular or Catholic [58]. With respect to secondary education, the core curriculum, including mathematics, is the same for everyone. However, under teachers' and parents' guidance, students elect to follow vocationally oriented, humanities oriented or classically oriented tracks, and there is a general understanding that these form an academic hierarchy [59]. Importantly, research has shown that school type and track influence significantly the mathematical learning of Flemish students [58, 60, 61, 62]. Thus, in relation to the UK, Flanders may be a more appropriate research site, particularly from the perspective of its multicultural demographic and plurality of educational provision.

Despite its having a very active and well-regarded mathematics education research community, particularly at the Catholic University of Leuven, little has been published to explain Flemish PISA success, and so it remains a site of untapped interest. However, the evidence of my own work [8, 9] has highlighted several key issues likely to underpin a secure learning of mathematics. Firstly, Flemish teachers place a very strong emphasis on students' acquisition of conceptual knowledge as the basis for later procedural skills. Secondly, in making frequent reference to links between and within topics, they attend explicitly to the broader structural relationships of mathematics. Thirdly, they regularly encourage students to engage, typically in the public domain, with mathematical reasoning. Fourthly, Flemish teachers constantly engage in high level questioning and the activation of their students' prior knowledge. Fifthly, they explain and coach in similar, high, proportions of time. The totality indicates not only a sophisticated pedagogical tradition but one in which conceptual and procedural knowledge are developed simultaneously, with the former being privileged as the basis for the latter. Interestingly, although this would have had little serious impact on Flemish students' performance on either TIMSS or PISA, there was little evidence of students being encouraged to solve non-routine problems.

By way of example, and to highlight differences between the Finnish and Flemish approaches, the following extract derives from the first of a sequence of lessons taught to a grade 5 (year 6) class on percentages. Emke, the teacher, began by inviting her students to share the percentage-related artefacts they had brought from home before asking questions to elicit the extent of their understanding of the relationships embedded in them. For example, in the context of a yoghurt pot containing nine percent fruit, she asked, *does this mean there are nine pieces of fruit in the yoghurt? Would it make a difference if the pot were larger or smaller?* For the remainder of the lesson she exploited base ten number blocks. In the first instance she asked students to take four hundreds and place five cubes in front of each hundred. A short discussion led to her writing, what she called the formulation, (*five* *for each 100) of 400* on the board. Several related tasks were completed similarly, including, for example, (8 for each hundred) of 450, before Emke shifted attention in various ways. Firstly, she invited students to construct concrete models for different formulations. Secondly, she asked students to stack their blocks rather than lay them side by side. That is, with formulations like (5 per 100) of 300 she now expected students to work with a stack rather than separately placed hundred squares. Thirdly, she presented various concrete representations from which students had to derive formulations. Fourthly, she drew representations, for example, three squares with two in front of each, and invited students to write the formulation. Finally, she offered, orally, examples like *two for every hundred of three hundred* and asked students to write both diagrams and the formulations. Throughout the lesson, which lasted around forty minutes, Emke's attention was on the conceptual basis of percentages. Each task was solved individually before being discussed collectively. The various shifts of attention reinforced this conceptual objective. Interestingly, her formulations, every one of which was written on the board, trailed the procedure she would introduce the following lesson, but for now procedures, while implicit in what she did, were subordinated to a deep conceptual knowledge. The underlying principle of mathematics teaching in Flanders seems to be not to rush too quickly into teaching procedures, but to allow the concepts to develop first.

Thus far, I have tried to show that an uncritical reliance on international tests of mathematical competence may not be the most helpful way of identifying appropriate sources of adaptive potential. In so doing I have highlighted what I believe are the inadequacies of both Singapore and Finland but leave Flanders open to further investigation. Importantly, underpinning this paper has been the inevitable concern, as someone committed to improving the quality of mathematics instruction in English schools, that what I may desire may conflict with, say, the aims of government. In this respect, I suspect that TIMSS is of more interest to me than PISA, while the converse may be true of politicians. My objective is to find ways of inducting as many students as possible into an intellectually worthwhile engagement with mathematics in ways that encourage and sustain students' interests in the subject beyond compulsory schooling. Governments, for reasons I still do not understand, seem focused on PISA's dubious outcomes − not least, I suspect, because the OECD asserts that PISA success implies economic growth. For example, shortly after publishing the results of PISA 2009, the OECD announced that

> "A modest goal of having all OECD countries boost their average PISA scores by 25 points over the next 20 years… implies an aggregate gain of OECD GDP of USD 115 trillion over the lifetime of the generation born in 2010... Bringing all countries up to the average performance of Finland, OECD's best performing education system in PISA, would result in gains in the order of USD 260 trillion" [63, p. 6]

Against such powerful, and seemingly unwarranted, arguments it is hard to sustain a thesis that the interests of mathematics are better served by TIMSS than PISA, and yet I think it is clear, students who succeed on the various TIMSS assessments are more likely to understand mathematics than those who succeed on PISA. This is an issue the Finns are slowly coming to accept. Indeed, as Tarvainen and Kivelä observe,

> "one has to consider the possibility that the first place in the PISA study is a Pyrrhic victory: are the Finnish basic schools stressing too much numerical problems of the type emphasized in the PISA study, and are other countries, instead, stressing algebra, thus guaranteeing a better foundation for mathematical studies in upper secondary schools and in universities and polytechnics" [37, p. 10].

Finally, in the same way that throughout my life, despite frequent explorations of other composers and their distinctive perspectives on music I always return to Bach, I return to my first professional love, Hungary. The didactical skill of Hungarian teachers, based on a profound understanding of mathematical concepts and their interrelations, is to mathematics what Bach, "the immortal god of harmony" (Beethoven, 1801 in [64]), is to music. It is peerless. I close by offering two examples of exquisite Hungarian teaching.

The first example, for which I offer thanks to Jenni Back, derives from a grade 1 class (Year 2) working on number sequences. It began with the teacher, Klara, writing the following on the board

3   7   6   10   _   _   _   _   _

Klara asked her children to read the numbers before asking them to identify the rule for the sequence. Students volunteered that the first jump was add four, the second was subtract one, add four and so on. The class then predicted, and Klara wrote in the spaces, that the next numbers in the sequence were 9, 13, 12, 16 and 15.

This was followed by a series of questions, each linked to one of the numbers of the sequence. For example, nine was the answer to *I am thinking of the largest one-digit number, who am I?* Having been given the correct answer, Klara wrote a capital 'I' above the number. Other questions related to the value of a particular Cuisenaire rod, and concepts like more than, two digit number, double, the sum of the digits, smaller neighbour, bigger neighbour, the difference of the digits. Each correct answer prompted Klara to write a capital letter above the corresponding number.

Finally, with all bar one number having been accompanied by a letter, Klara asked her students to suggest properties for 10. Their propositions included

It is the bigger neighbour of 9.
It is the smallest 2-digit number.
It is the smaller neighbour of 11.
The sum of its digits is 1.
It's an even number.
It is the sum of the 1 and 9.

Klara: Yes, it is the sum of the 1 and 9… and who knows the letter in my hand?

Chorus: SZ*

Klara: Yes, so where are we going today?

Chorus: Bábszínház (puppet theatre)

Each time I watch this clip, and I have on several occasions, I smile. I have watched many Hungarian lessons and smiled. But I have rarely done the same in England classrooms, and this makes me sad. When I have smiled, as in the cloakroom ticket episode above, it has been for all the wrong reasons. This episode, it seems to me, was a delightfully self-contained sequence in which Klara exploited a very simple starting point to the full. Having identified the rules and completed the missing numbers she continued to use them in a variety of ways within the larger narrative of the lesson – a narrative that can be construed both literally and figuratively. Almost every child in Klara's class was publicly involved. Like the triangular problem discussed at the start of this paper, it embodied many concepts, each of which would have been visited regularly during a sequence of lessons in which Klara only ever exposed her students to integers up to twenty. There was no expectation that they should learn algorithms before they had acquired a well-developed 'number sense' [65, 66, 67]. There were no lesson objectives on the board, although the lesson was clearly driven by many, and there were no success criteria. There was no interactive whiteboard obstructing the exploitation of the most important resources in any classroom – the subject matter and pedagogic content knowledge of the teacher and the engagement of her children. In this respect both teacher and children seemed happy to be participants in this emergent mathematical community.

The second example derives from the second lesson of a sequence on the teaching of linear equations reported by Andrews and Sayers [68]. The teacher, Emese, had posed a problem concerning the delivery of potatoes to the school's kitchen. It went: *On two consecutive days the same weight of potatoes was delivered to the school's kitchen. On the first day 3 large bags and 2 bags of 10kg were delivered. On the second day 2 large bags and 7 bags of 10kg were delivered. If the weight of each large bag was the same, what was the weight of the large bag?* A discussion ensued from which it emerged that an equation could be constructed and that x kg would represent the weight of the large bag. Shortly after this a volunteer wrote $3x + 20 = 2x + 70$ on the board. Next, having established by means of a series of questions that intuitive strategies were now insufficient, Emese drew a picture of a scale balance with the various bags represented on both sides. Drawing on a student's suggestion she erased two small bags from each side, leaving a representation of $3x = 2x + 5$. Next she erased two large bags from each side to show one large bag balancing 5 small. Then, in response to her request, students volunteered sufficient for her to write

* In Hungarian the juxtaposition of s and z in this manner constitutes an alphabetic letter linked to the sound of s in the English word sit.

alongside her drawings:

$$\begin{aligned} 3x + 20 &= 2x + 70 \qquad \vert -20 \\ 3x &= 2x + 50 \qquad \vert -2x \\ x &= 50. \end{aligned}$$

Finally, Emese reminded her class of the importance of checking and did so.

I have always regarded this was an extremely elegant and didactically powerful piece of teaching. Admittedly, there is some scepticism in the mathematics education research community regarding the efficacy of the balance scale as an embodiment suitable for equations, typically drawing on the argument that it cannot adequately represent negative amounts or is unfamiliar to students used to electronic scales [69]. However, I think that the ways in which Emese and her students managed this and later episodes dismiss both criticisms. So why is it didactically powerful? Firstly, Emese built her exposition on a word problem that gave context to the algebra she intended to introduce. No-one in the class seriously thought she was attempting to model reality; it was a realistic problem in the tradition of the Dutch realistic mathematics education [70]. That is, it was an 'imaginably real' problem designed to scaffold students' access to powerful mathematical ideas. Secondly, having previously rehearsed mental solutions to equations with the unknown on one side of the equals sign, she did not attempt to teach formal approaches to such equations but highlighted the need to do so with an equation, due to its having the unknown on both sides, too difficult to solve by intuitive methods. In other words, she provided a clear warrant for her formal exposition. Thirdly, the ways in which she linked the various representations further scaffolded students' access to the ideas being introduced. In this respect, the removal of the same number of bags from both sides alluded very clearly to the principle of doing the same to both sides. Fourthly, she reminded her class of the need to check the result obtained against the original wording of the problem. Fifthly, the whole episodes lasted a few seconds longer than five minutes. It was very efficiently managed but never felt rushed.

Interestingly, this lesson was videotaped more than a decade after I had observed a lesson taught, on the same topic, by a different teacher in a different school [71]. The only difference was that he actually brought a set of scales into the classroom and made even more explicit the link between what Bruner [72] described as the enactive, iconic and symbolic representations. Such similarities bring me to another important issue to be considered when examining the adaptive potential of one country's practices for another's. Hungarian teachers operate within an accepted and collectively understood didactical tradition. There is little sense that individuals teach a topic in idiosyncratic ways they think will work, as is commonplace in England, because they understand why they do things in the ways they do and, importantly, acknowledge their collective responsibility to those teachers who inherit their students in subsequent grades. Can UK governments afford to continue to avoid a prescribed, assuming it is appropriately warranted, pedagogy?

In closing I offer some thoughts about what a good curriculum might entail and how it may be taught. They derive not only from the material presented above but from twenty years of observing classrooms and scrutinising the literature pertaining to effective mathematics teaching in different cultural contexts. They are presented without commentary, as I believe they speak for themselves and offer a strong steer to policy makers in this country, should they have the courage to grapple with them. However, one issue, missing in all the above, is that the UK constantly steals its children's childhood. The Finns do not send their students to school until they are seven and do not start teaching formal mathematics until they are eight. This is not unique; no other European system expects children to start school, not formal mathematics but school, before they are six, let alone five. There is, it seems to me, a perverse irony that successive UK governments have seen earlier and more formal interventions as the solution to UK underachievement. This really is one issue where political and popular rhetoric, despite their convergence, defy all the evidence.

A good curriculum expects

- Mathematics to be built on sound conceptual foundations.
- Students to acquire a secure number sense before introducing arithmetic.
- Mathematics to be difficult.
- Mathematics to be a problem-solving activity.
- Problems to exemplify generality and problematise particularity.
- Teachers to develop rather than state concepts and procedures.
- Teachers to make connections within and between topics.
- Students to engage with proof and justification.
- Teachers to encourage mathematical vocabulary.
- Applications to be subordinated to mathematics itself.
- Mathematical ideas to be revisited constantly within the problems offered.

And pedagogically

- Teachers expect to teach each class as a unit – within class differentiation is an alien concept in most European systems.
- Teachers take their time; in many countries the fewer problems solved during a lesson the better, it means they have been done properly and with understanding.
- Teachers talk, or manage the talk of others, for the majority of a lesson.

- Students are expected to operate in a public domain.
- Students spend relatively little time working alone.
- Routine exercises are few - teachers present a few carefully selected and challenging problems that incorporate systematic variation from one to the next.
- Homework is frequent but short and bridges successive lessons.

If government is serious in its attempts to reform the mathematics curriculum then it will need to account not only for the matters discussed above but also the evidence that teachers work within three curricula [73]. They work within an intended curriculum reflecting the knowledge and skills privileged by the system in which they operate; they work within a received curriculum, amenable only to inference, reflecting the hidden and culturally derived beliefs and practices teachers acquire by dint of being who they are; and they work within an idealised curriculum, which is articulable, and reflects individual teachers' personal and experientially informed beliefs. Reform will fail if attention is paid only to the intended curriculum. The long-standing and collective practices of the received curriculum are slow to change, as highlighted above in the disappointing espoused beliefs of many English teachers. Any serious reform will need to accept that overcoming such deep-seated and largely counter-productive beliefs will be an expensive and long term enterprise – sending selected individuals for a few days' training in the expectation that they will, to use that awful word, 'cascade' to colleagues is naïve and, essentially, a waste of public money. Finally, teachers will need to be persuaded that proposed changes can be made idealised. That is, until teachers see value in the proposals and can find ways to integrate them into existing articulable belief and practice frameworks, curricular intentions may be prone to subversion.

References

1. F. Leung, The mathematics classroom in Beijing, Hong Kong and London. *Educational Studies in Mathematics*, (1995) **29** (4), pp. 297-325.
2. G. Kaiser, Educational philosophies and their influence on mathematics education – an ethnographic study in English and German mathematics classrooms. *Zentralblatt fur Didaktik der Mathematik*, (2002) **34** (6), pp. 241-257.
3. G. Kaiser, K. Hino & C. Knipping, Proposal for a framework to analyse mathematics education traditions in eastern and western traditions. In F. Leung, K.-D. Graf & F. Lopez-Real (Eds.), *Mathematics education in different cultural traditions: A comparative study of East Asia and the West*, the 13th ICMI Study (pp. 319-351). New York: Springer (2006).

4. L. Haggarty & B. Pepin, An investigation of mathematics textbooks and their use in English, French and German classrooms: who gets an opportunity to learn what? *British Educational Research Journal*, (2002) **28** (4), pp. 567-590.
5. B. Pepin & L. Haggarty, Mathematics textbooks and their use in English, French and German classrooms: a way to understand teaching and learning cultures. *Zentralblatt fur Didaktik der Mathematik*, (2001) **33** (5), pp. 158-175.
6. P. Andrews, The curricular importance of mathematics: A comparison of English and Hungarian teachers' espoused beliefs. *Journal of Curriculum Studies*, (2007) **39** (3), pp. 317-338.
7. P. Andrews, A comparison of Hungarian and English mathematics teachers' professional goals: Manifestations of implicit cultural expectations. In A. Gagatsis, T. Rowland, A. Panaoura & A. Stylianides (Eds.), *Mathematics education research at the University of Cyprus and the University of Cambridge: A symposium* (pp. 5-20). Lefkosia: School of Social Sciences and Sciences of Education, the University of Cyprus (2010).
8. P. Andrews, Comparative studies of mathematics teachers' observable learning objectives: validating low inference codes. *Educational Studies in Mathematics*, (2009) **71**(2), pp. 97-122.
9. P. Andrews, Mathematics teachers' didactic strategies: Examining the comparative potential of low inference generic descriptors. *Comparative Education Review* (2009) **53**(4), pp. 559-581.
10. D. Clarke, The LPS research design. In D. Clarke, C. Keitel & Y. Shimizu (Eds.), *Mathematics classrooms in twelve countries: The insider's perspective* Rotterdam: Sense Publishers (2006) pp. 15-36.
11. Z. Usiskin, Misidentifying factors underlying Singapore's high test scores: A strong curriculum is not the sole reason for Singaporean students’ success on international assessments. *Mathematics Teacher*, (2012) **105**(9), pp. 666-670.
12. L. Fan & Y. Zhu, From convergence to divergence: the development of mathematical problem solving in research, curriculum, and classroom practice in Singapore. *Zentralblatt fur Didaktik der Mathematik*, (2007) **39**(5/6), pp. 491-501.
13. R. Laukkanen, Finnish strategy for high-level education for all. In N. C. Soguel & P. Jaccard (Eds.), *Governance and Performance of Education Systems* (2008) Dordrecht: Springer (pp. 305–324).
14. Office for Standards in Education. *The education of six year olds in England, Denmark and Finland: An international comparative study*. (2003). London: Ofsted.
15. Office for Standards in Education. Finnish pupils’ success in mathematics: Factors that contribute to Finnish pupils’ success in mathematics (2010). Manchester: Ofsted.

16. P. Sahlberg, The fourth way of Finland. *Journal of Educational Change*, (2011) **12**(2), pp. 173-185.
17. J. Välijärvi, The system and how does it work: Some curricular and pedagogical characteristics of the Finnish comprehensive school. *Education Journal*, (2004) **32**, pp. 31-55.
18. S. Lie, P. Linnakylä & A. Roe,. Northern lights on PISA. In S. Lie, P. Linnakylä & A. Roe (Eds.), *Northern Lights on PISA: Unity and diversity in the Nordic countries in PISA 2000* (2003). Oslo: Department of Teacher Education and School Development, University of Oslo (pp. 7-20).
19. A. Antikainen, In search of the Nordic model in education. *Scandinavian Journal of Educational Research*, (2006) **50**(3), pp. 229-243.
20. P. Reinikainen, Amazing PISA results in Finnish comprehensive schools. In H. Niemi, A. Toom & A. Kallioniemi (Eds.), *Miracle of education: The principles and practices of teaching and Learning in Finnish schools* (2012). Rotterdam: Sense Publishers (date?) pp. 3-18.
21. I. Halinen & R. Järvinen, Towards inclusive education: the case of Finland. *Prospects*, (2008) **38**(1), pp. 77-97.
22. E. Aho, K. Pitkänen & P. Sahlberg, *Policy development and reform principles of basic and secondary education in Finland since 1968*. Washington: The World Bank. (2006).
23. P. Sahlberg, Education policies for raising student learning: the Finnish approach. *Journal of Education Policy*, (2007) **22**(2), pp. 147-171.
24. J. Poikolainen, A case study of parents' school choice strategies in a Finnish urban context. *European Educational Research Journal*, (2012) **11**(1), pp. 127-144.
25. H. Savolainen, Responding to diversity and striving for excellence: The case of Finland. *Prospects*, (2009) **39**(3), pp. 281-292.
26. R. S. Hausstätter & M. Takala, Can special education make a difference? Exploring the differences of special educational systems between Finland and Norway in relation to the PISA results. *Scandinavian Journal of Disability Research*, (2011) **13**(4), pp. 271-281.
27. J. Kivirauma & K. Ruoho, Excellence through special education? Lessons from the Finnish school reform. *International Review of Education*, (2007) **53**, pp. 283-302.
28. P. Vilenius-Tuohimaa, K. Aunola & J.-E. Nurmi, The association between mathematical word problems and reading comprehension. *Educational Psychology*, (2008) **28**, pp. 409–426.
29. H. Simola, The Finnish miracle of PISA: Historical and sociological remarks on teaching and teacher education. *Comparative Education*, (2005) **41**, pp. 455-470.

30. J. Tuovinen, Learning the craft of teaching and learning from world's best practice. The Case of Finland. In D. McInerney & A. Liem (Eds.), *Teaching and learning: International best practice* (2008). Charlotte, NC: Information Age Publishing, pp. 51-77.
31. H. Niemi & R. Jakku-Sihvonen, Research-based teacher education. In R. Jakku-Sihvonen & H. Niemi (Eds.), *Research-based teacher education in Finland* (2006). Turku: Finnish Educational Research Association, pp. 31-50.
32. R. Jyrhämä, H. Kynäslahti, L. Krokfors, R. Byman, K. Maaranen, A. Toom & P. Kansanen, The appreciation and realisation of research-based teacher education: Finnish students' experiences of teacher education. *European Journal of Teacher Education*, (2008) **31**(1), pp. 1 - 16.
33. H. Niemi, The societal factors contributing to education and schooling in Finland. In H. Niemi, A. Toom & A. Kallioniemi (Eds.), *Miracle of education: The principles and practices of teaching and Learning in Finnish schools* (2012). Rotterdam: Sense Publishers (2012) pp. 19-38.
34. X. Liang, Assessment use, self-efficacy and mathematics achievement: comparative analysis of PISA 2003 data of Finland, Canada and the USA. *Evaluation & Research in Education*, (2010) **23**(3), pp. 213-229.
35. A. Schleicher, Securing quality and equity in education: Lessons from PISA. *Prospects*, (2009) **39**(3), pp. 251-263.
36. K. Astala, S. Kivelä, P. Koskela, O. Martio, M. Näätänen & K. Tarvainen, The PISA survey tells only a partial truth of Finnish children's mathematical skills. *Matilde*, (2006) **29,** p. 9.
37. K. Tarvainen & S. Kivelä, Severe shortcomings in Finnish mathematics skills. *Matilde,* (2006) **29,** p. 10.
38. G. Malaty, Mathematics and mathematics education development in Finland: the impact of curriculum changes on IEA, IMO and PISA results. Paper presented at the *Proceedings of the 10th International Conference of the Mathematics Education into the 21st Century Project*, Dresden University of Applied Sciences (2010).
39. R. Adams, Response to 'Cautions on OECD's recent educational survey (PISA). *Oxford Review of Education*, (2003) **29**(3), pp. 377-389.
40. P. Linnakylä, Reading in Finland. In C. Papanastasiou & V. Froese (Eds.), *Reading Literacy in 14 Countries*. Lefkosia: University of Cyprus Press (2002).
41. D. Wallgren, Swedish-speaking Finns have strong presence in major corporations, *Helsingin Sanomat - International Edition* (2011, August 28).
42. J. Karhunen & M. Keloharju, Shareownership in Finland 2000. *Liiketaloudellinen Aikakauskirja* (The Finnish Journal of Business Economics) (2001) **2,** pp. 188-226.
43. S. Kupiainen, J. Hautamäki & T. Karjalainen, The Finnish education system and PISA: University of Helsinki and the Finland Ministry of Education, Finland (2009).

44. E. Niemi & J. Metsämuuronen, (Eds.), Miten Matematiikan Kehittyvät?: *Matematiikan oppimistulokset peruskoulun viidennen vuosiluokan jälkeen vuonna 2008.* Helsinki: Edita Prima Oy (2010).
45. I. Carlgren, K. Klette, S, Mýrdal, K, Schnack & A. Simola, Changes in Nordic teaching practices: From individualised teaching to the teaching of individuals. *Scandinavian Journal of Educational Research,* (2006) **50,** pp. 301–326.
46. P. Kupari, Recent developments in Finnish mathematics education. *Nordic Studies in Mathematics Education,* (2004) **9,** pp. 7-20.
47. N. Norris, R. Asplund, B. MacDonald, J. Schostak & B. Zamorski, *An independent evaluation of comprehensive curriculum reform in Finland*. Helsinki: National Board of Education (1996).
48. L. Savola, (2010). Comparison of the classroom practices of Finnish and Icelandic mathematics teachers. *Journal of Mathematics Education at Teachers College* (Fall-Winter 2010) pp. 7-13.
49. E. Pehkonen, How Finns learn mathematics: What is the influence of 25 years of research in mathematics education? In M. Lepis (Ed.), *Teaching mathematics: Retrospectives and perspectives* (pp. 71-101). Tallinn University: Institute of Mathematics and Natural Sciences (2009).
50. A. Ryve, K. Hemmi & M. Börjesson, Discourses about School-based Mathematics Teacher Education in Finland and Sweden. *Scandinavian Journal of Educational Research*, (2011) pp. 1-16.
51. P. Andrews, Finnish mathematics teaching: a case of uniquely implicit didactics. In T. Dooley, D. Corcoran & M. Ryan (Eds.), *Keynote presentation to the Fourth Conference on Research in Mathematics Education* (2011). Dublin: St Patrick's College, pp. 3-18.
52. P. Andrews, Finnish mathematics teaching from a reform perspective: A video-based case study analysis. *Comparative Education Review,* **57** (2), (2013)
53. P. Andrews, A. Ryve, K. Hemmi & J. Sayers, Finnish mathematics teaching and PISA success: An enigma in search of an explanation. *Educational Studies in Mathematics*, (2012).
54. I. De Meyer, H. De Vos & L. Van de Poele, *Worldwide learning at age 15: First results from PISA 2000.* Gent: University of Gent and Ministry of the Flemish Community Education Department (2002).
55. I. De Meyer, J. Pauly & L. Van de Poele, *Learning for Tomorrow's Problems: First Results from PISA 2003*. Gent: University of Gent and the Ministry of Education of the Flemish Community of Belgium (2005).
56. I. De Meyer, *Science competencies for the future in Flanders: The first results from PISA 2006.* Gent: University of Gent and the Ministry of Education of the Flemish Community of Belgium (2008).

57. I. De Meyer & N. Warlop, Leesvaardigheid van 15-jarigen in *Vlaanderen: De eerste resultaten van PISA 2009.* Gent: Universiteit Gent and Departement Onderwijs and Vorming (2010).
58. M.-C. Opdenakker, J. Van Damme, B. De Fraine, G. Van Landeghem & P. Onghena, *The Effect of schools and classes on mathematics achievement. School Effectiveness and School Improvement*, (2002) **13**(4), pp. 399-427
59. P. Op 't Eynde, E. De Corte & L. Verschaffel, Beliefs and metacognition: An analysis of junior-high students' mathematics-related beliefs. In M. Veenman & A. Desoete (Eds.), *Metacognition in mathematics education* (2006) New York: Nova Science (pp. 83-101).
60. H. Brutsaert, Home and school influences on academic performance: state and Catholic elementary schools in Belgium compared. *Educational Review*, (1998) **50**(1), pp. 37-44.
61. G. Pugh & S. Telhaj, Faith schools, social capital and academic attainment: evidence from TIMSS-R mathematics scores in Flemish secondary schools. *British Educational Research Journal*, (2007) **34**(2), pp. 235-267.
62. H. Pustjens, E. Van de Gaer, J. Van Damme, P. Onghena & G. Van Landeghem, The short-term and the long-term effect of primary schools and classes on mathematics and language achievement scores. *British Educational Research Journal*, (2007) **33**(3), pp. 419-440.
63. Organisation for Economic Cooperation and Development. *The high cost of low educational performance: The long-run economic impact of improving PISA outcomes*. Paris: OECD PISA (2010).
64. M. Hamburger, (Ed.). *Beethoven: Letters, journals and correspondence*. London: Thames and Hudson (1992).
65. B. Casey, J. Kersh & J. Young, Storytelling sagas: an effective medium for teaching early childhood mathematics. *Early Childhood Research Quarterly*, (2004) **19**(1), pp. 167-172.
66. S. Howell & C. Kemp, Defining early number sense: A participatory Australian study. *Educational Psychology*, (2005) **25**(5), pp. 555-571.
67. D.-C. Yang & M.-N. Li, An investigation of 3rd-grade Taiwanese students' performance in number sense. *Educational Studies*, (2008). **34**(5), pp. 443-455.
68. P. Andrews & J. Sayers, Exemplary teaching of linear equations in Finland, Flanders and Hungary. *Journal of Mathematical Behavior*, (In press) (2012).
69. S. Pirie, & L. Martin, The equation, the whole equation and nothing but the equation! One approach to the teaching of linear equations. *Educational Studies in Mathematics*, (1997) **34** (29), pp. 159-181.

70. M. Van den Heuvel-Panhuizen, The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics*, (2003) **54**(1), pp. 9-35.
71. P. Andrews, Opportunities to learn in the Budapest mathematics classroom. *International Journal of Science and Mathematics Education*, (2003) **1** (2), pp. 201-225.
72. J. Bruner, *Towards a theory of instruction*. Cambridge, Mass.: Harvard University Press (1966).
73. P. Andrews, The cultural location of teachers' mathematical knowledge: Another hidden variable in mathematics education research? In T. Rowland & K. Ruthven (Eds.), *Mathematical knowledge in teaching* (2011b) New York: Springer (pp. 99-118).

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