Mathematical fluency *without* exercises: Mathematical Etudes

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MATHEMATICAL FLUENCY *WITHOUT* DRILL AND PRACTICE

Colin Foster asks how can we avoid letting 'practice' dominate the teaching of the new mathematics national curriculum

Introduction

The word 'practice' appears twice in the short 'Aims' section of the *KS3 Programme of study* (DfE, 2013). The first stated aim is that all pupils:

*… become **fluent** in the fundamentals of mathematics, including through varied and frequent practice with increasingly complex problems over time, so that pupils develop conceptual understanding and the ability to recall and apply knowledge rapidly and accurately*. (p. 2)

This optimistic sentence implies that focusing on fluency will lead eventually to conceptual understanding and confidence in applying the knowledge gained. This reminds me of John Holt's

I very much agree with. However, the following sentence, that '*Those who are not sufficiently fluent should consolidate their understanding, including through additional practice, before moving on*', sounds to me like a recipe for never-ending, low-level, imitative rehearsing of knowledge and skills until students earn the right to anything more stimulating.

It is easy to see how students can become trapped in tedious, repetitive work, endlessly 'practising the finished product' (Prestage and Perks, 2006). Teachers are going to be told that certain students 'need more practice on X' before they are 'ready' to move on. Students will be discouraged and demotivated by constant, unimaginative repetition

Foster, C. (2014). Mathematical fluency without drill and practice. *Mathematics Teaching, 240*, 5–7

Mathematics Lessons

Tedious Exercises

What are the factors of 16?

Lovely Rich Tasks

Find some numbers with exactly 5 factors.

Procedural Fluency

Knowing when and how to apply a mathematical procedure and being able to perform it "accurately, efficiently, and flexibly" (NCTM, 2014, p. 1).

"The national curriculum for mathematics aims to ensure that all pupils:

- become ***fluent*** in the fundamentals of mathematics…
- ***reason mathematically***…
- can ***solve problems***…"

DfE (2013, p. 2, original emphasis)

Procedural Fluency

"It is a profoundly erroneous truism … that we should cultivate the habit of thinking what we are doing. The precise opposite is the case. ~~Civilization~~ Mathematics advances by extending the number of important operations which we can perform without thinking about them."

(Whitehead, 1911, 58-61).

Internalising procedures

Give opportunities for learners to develop their fluency in important mathematical procedures while something “a bit more interesting” is going on.

Subordinating the skill

“practice can take place without the need for what is to be practised to become the focus of attention” (p. 34)

Hewitt (1996)

Procedural Fluency

Is it only possible to achieve it by subjecting pupils to dull, repetitive exercises?

“the notion that if a child repeats a meaningless statement or process enough times it will become meaningful is as absurd as the notion that if a parrot imitates human speech long enough it will know what it is talking about”

(Holt, 1990, p. 193)

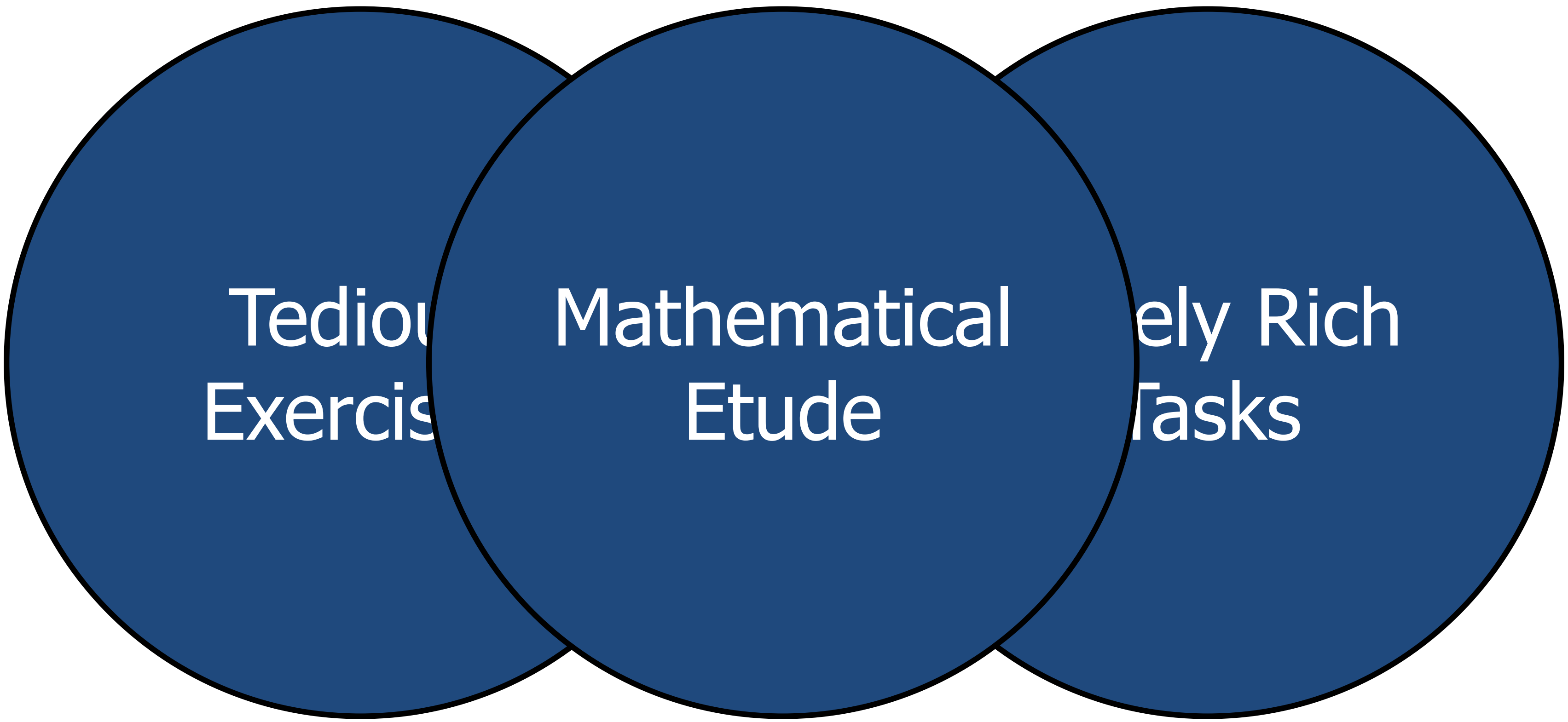
The problem with "rich tasks"

Rich
Open-ended
Exploratory
Investigative
Problem-solving

Let the fluency develop incidentally while students work on more interesting problems?

- Students take multiple approaches and learn different things
- Possibility of avoiding less comfortable areas (play to strengths, use what they know)
- Lack of concentrated focus on any one particular technique

Mathematical Etudes



Musical Étude

"originally a study or technical exercise, later a complete and musically intelligible composition exploring a particular technical problem in an esthetically satisfying manner"

Encyclopaedia Britannica

www.mathematicaletudes.com

The **Mathematical Etudes Project** aims to find creative, imaginative and thought-provoking ways to help learners of mathematics develop their fluency in important mathematical procedures.

Procedural fluency involves knowing when and how to apply a procedure and being able to perform it "accurately, efficiently, and flexibly" (NCTM, 2014, p. 1). Fluency in important mathematical procedures is a critical goal within the learning of school mathematics, as security with fundamental procedures offers pupils increased power to explore more complicated mathematics at a conceptual level (Foster, 2013, 2014, 2015; Gardiner, 2014; NCTM, 2014). The new national curriculum for mathematics in England emphasises procedural fluency as the first stated aim (DfE, 2013).

But it is often assumed that the only way to get good at standard procedures is to drill and practise them *ad nauseum* using dry, uninspiring exercises.

The **Mathematical Etudes Project** aims to find practical classroom tasks which embed extensive practice of important mathematical procedures within more stimulating, rich problem-solving contexts (Foster, 2011, 2013, 2014). For more details see the papers listed below or scroll down for some example tasks.

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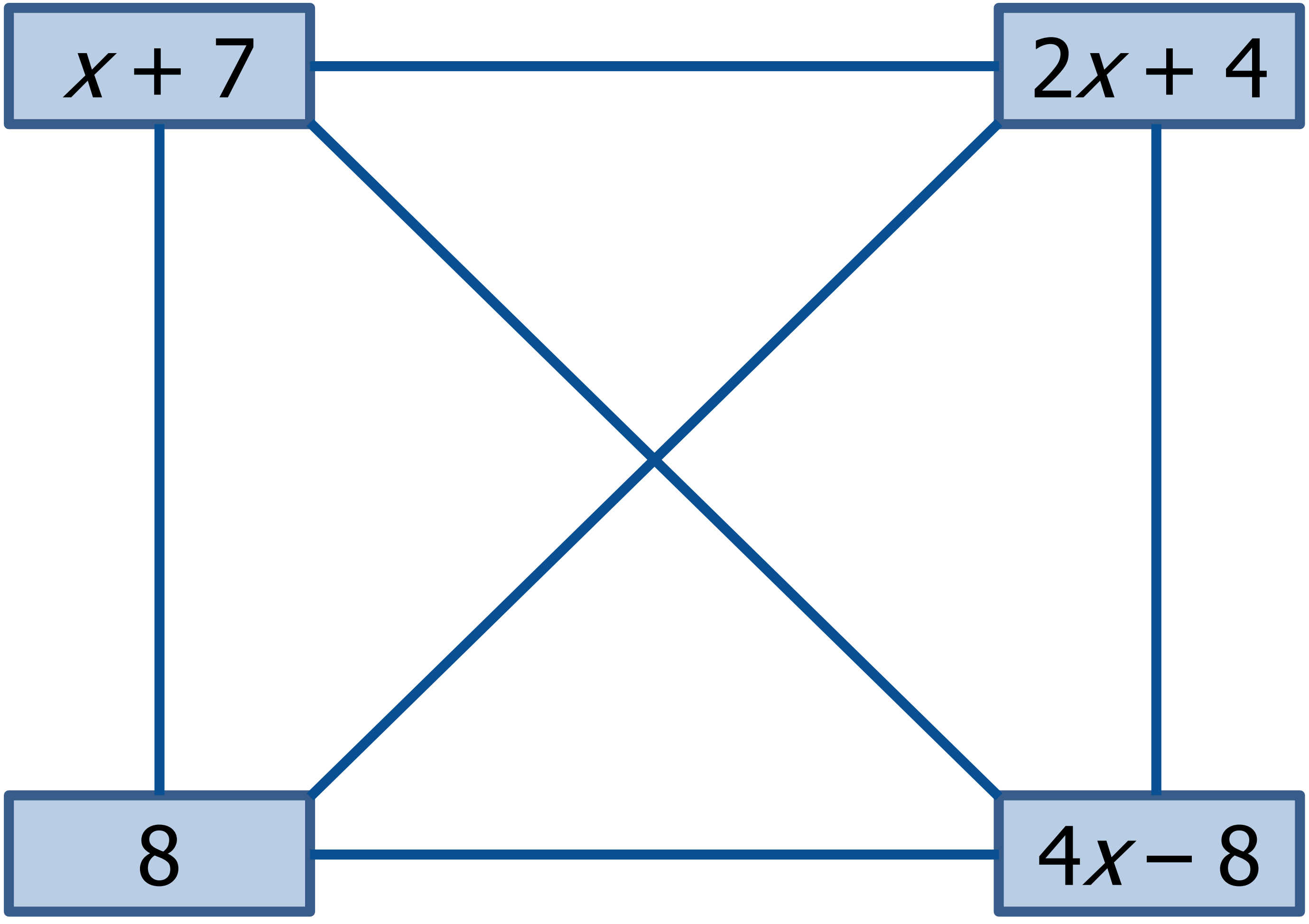
Foster, C. (2013). Mathematical études: Embedding opportunities for developing procedural fluency within rich mathematical contexts. *International Journal of Mathematical Education in Science and Technology, 44*(5), 765–774.



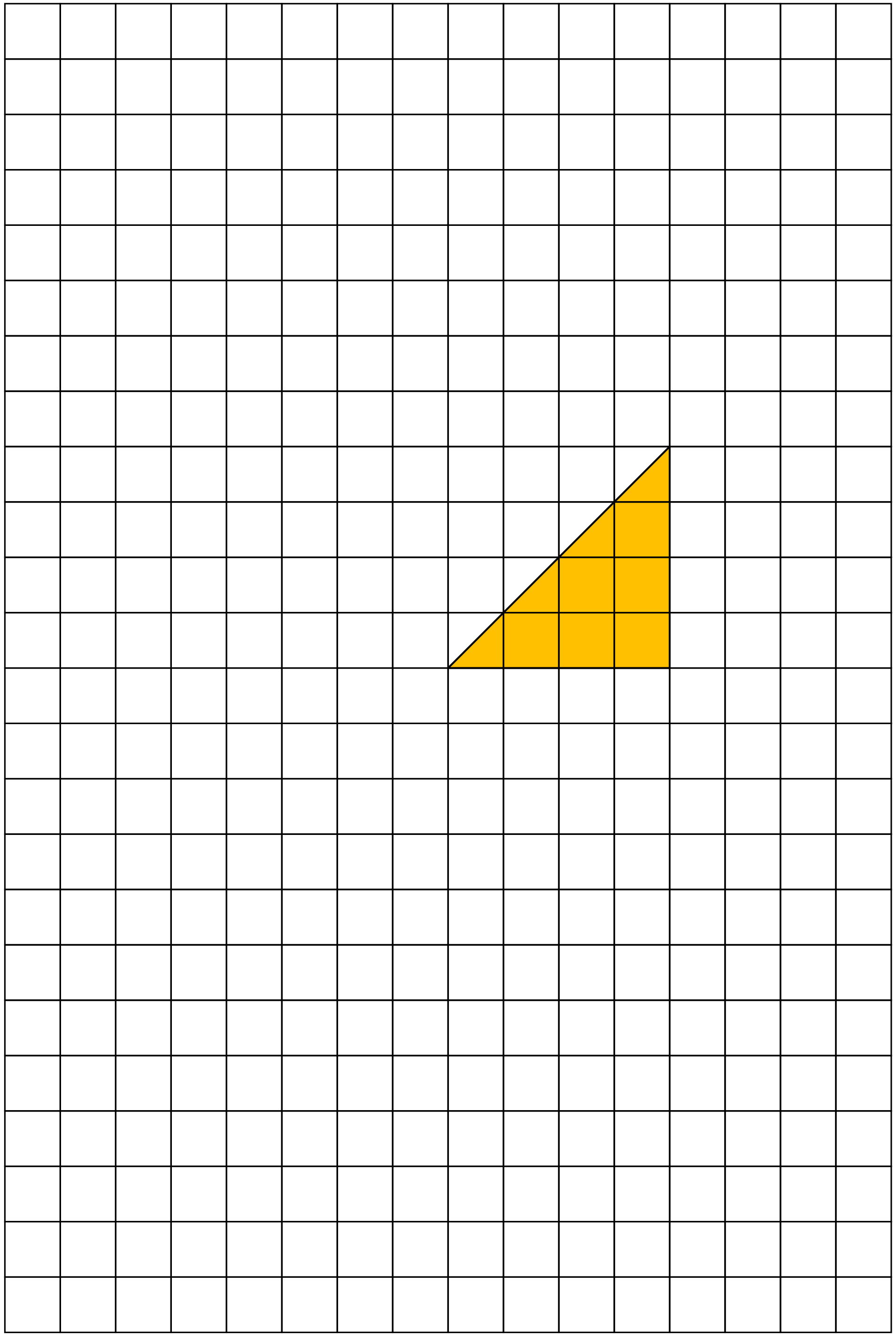
Foster, C. (2012). Connected expressions. *Mathematics in School, 41*(5), 32–33.

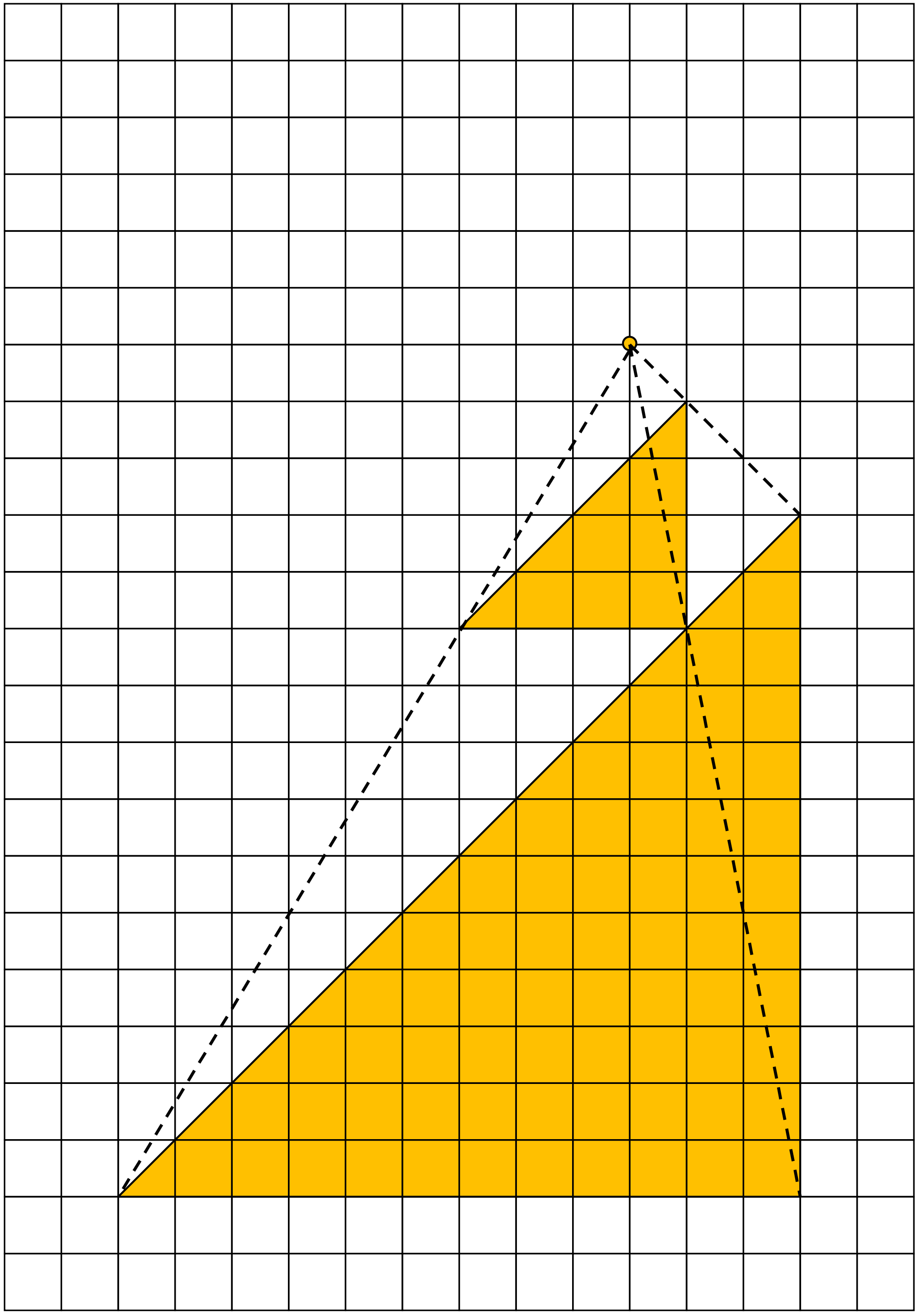
Enlargement drawings

Given an A4 piece of paper and a given shape and a given scale factor of enlargement, where can the centre of enlargement be so that all of the shape stays on the paper?

What is the locus of possible centres of enlargement for the triangle on the sheet if the scale factor is 3?

Foster, C. (2012). Working without a safety net. *The Australian Mathematics Teacher, 68*(2), 25–29.
Foster, C. (2013). Staying on the page. *Teach Secondary*, *3*(1), 57-59.





Solving Equations

Solve these equations.

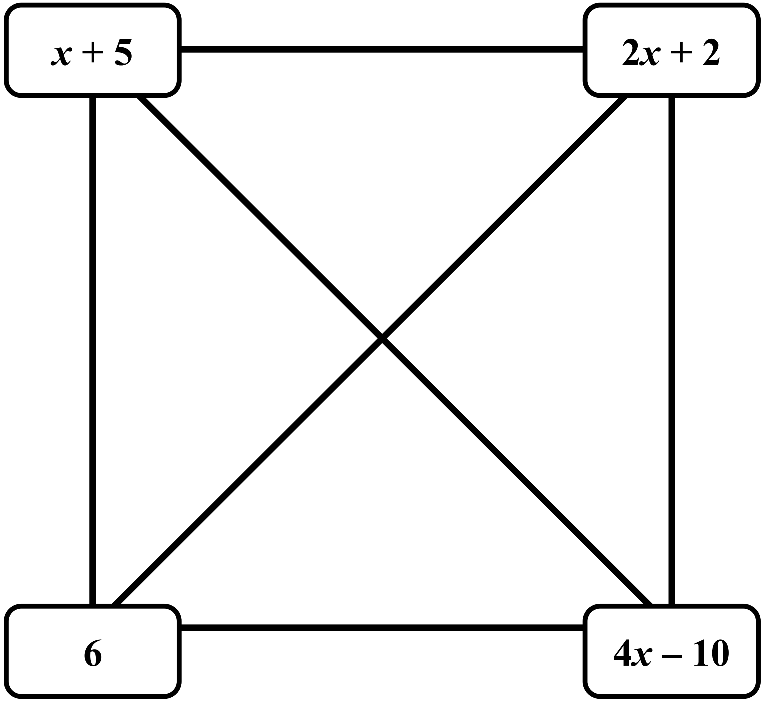
Show your method for each one.

1	$2x + 4 = 3x + 1$	**16**	$7x - 3 = 2x + 2$
2	$3x + 5 = 4x + 3$	**17**	$3x - 5 = x + 1$
3	$4x + 3 = 2x + 5$	**18**	$x + 6 = 2x - 5$
4	$2x - 3 = x - 1$	**19**	$3x - 4 = x - 6$
5	$2x + 1 = 3x - 2$	**20**	$3x + 9 = x - 5$
6	$5x - 3 = 2x + 12$	**21**	$6x - 4 = x + 16$
7	$4x + 9 = 8x - 31$	**22**	$x - 7 = 7x - 25$
8	$2x + 40 = 12x - 110$	**23**	$x + 5 = 4x - 4$
9	$3x + 4 = 5x - 8$	**24**	$6x + 5 = 3x - 7$
10	$2x - 8 = 3x - 16$	**25**	$x + 1 = 7x - 17$
11	$x + 1 = 5x + 9$	**26**	$3x - 4 = 5x + 6$
12	$5x = 2x + 12$	**27**	$8x + 3 = 6x + 15$
13	$9x + 8 = 20 - 3x$	**28**	$x = 20 - x$
14	$5x - 2 = x + 2$	**29**	$3x - 1 = x + 7$
15	$4x + 2 = 3x + 9$	**30**	$x - 6 = 9 - 2x$

Expression Polygons

In the diagram below, every line creates an equation.

So, for example, the line at the top gives the equation $x + 5 = 2x + 2$.



1. Write down and solve the six equations in this diagram.

2. What do you notice about your six solutions?

3. Now make up another diagram like this containing different expressions. Try to make the solutions to your *expression polygon* a "nice" set of numbers.

4. Make up some more *expression polygons* like this and see if other people can solve them.

Conclusion

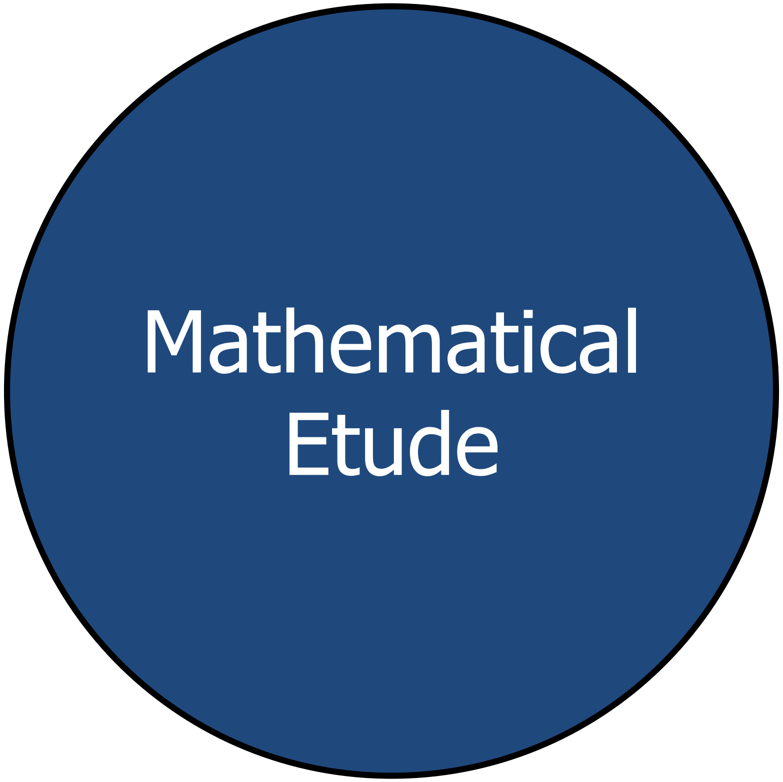
Even if all you care about is that students develop lots of procedural fluency ...

... you might as well use etudes!

And they are likely to have other (harder-to-measure) benefits too:

- Creative investigative inquiry
- Surprise and motivation
- Communication
- ...

Mathematical Etudes



MacGuffin

“Well, it’s the device, the gimmick, if you will, or the paper that the spies are after ... It doesn’t matter what it is ... The only thing that really matters is that in the picture the plans, documents, or secrets must seem of vital importance to the characters. To me, the narrator, they’re of no importance whatsoever.”

Alfred Hitchcock

Wiliam, D. (1997, September). Relevance as MacGuffin in mathematics education. In *British Educational Research Association Conference*, York, September 1997.

$5x + 8y$

Choose two of these expressions:

$x + y$	$x + 2y$	$x - 2y$	$x + 4y$	$2x + 3y$

Put them into the brackets below, with numbers in front of each:

$$\square(\qquad\qquad) \pm \square(\qquad\qquad)$$

$5x + 8y$

Choose two of these expressions:

$x + y$

$x - 2y$

$2x + 3y$

Put them into the brackets below, with numbers in front of each:

$$3(x + 2y) - 2(x + 4y)$$

$$\mathbf{5x + 8y}$$

Choose two of these expressions:

$x + y$ $\qquad$ $x - 2y$ $\qquad$ $2x + 3y$

Put them into the brackets below, with numbers in front of each:

$$3(\ x + 2y\ ) - 2(\ x + 4y\ )$$

$$= 3x + 6y - 2x - 8y$$

$$5x + 8y$$

Choose two of these expressions:

$x + y$ $\quad$ $x - 2y$ $\quad$ $2x + 3y$

Put them into the brackets below, with numbers in front of each:

$$3(\ x + 2y\ ) - 2(\ x + 4y\ )$$

$$= 3x + 6y - 2x - 8y$$
$$= x - 2y$$

$5x + 8y$

Here are your five expressions:

$x + y$	$x + 2y$	$x - 2y$	$x + 4y$	$2x + 3y$

- Can you find a way to make $5x + 8y$ using **two** different expressions?
- Can you find a way to make $5x + 8y$ using **more than two** different expressions?
- Can you find a way to make $5x + 8y$ using **all five** expressions?

Foster, C. (2016). The simple life. *Teach Secondary, 5*(2), 31–33.

Fractions

$$\frac{1}{6} \qquad \frac{1}{25} \qquad \frac{3}{5} \qquad \frac{3}{20} \qquad \frac{4}{15} \qquad \frac{5}{8}$$

Add up as many of these fractions as you like. You can't use any of them more than once.

Try to get a total as near to 1 as possible.

Is this a good set of fractions to use for this puzzle? Can you invent a better set?

Foster, C. (2014). Sum fractions. *Teach Secondary, 3*(5), 48–49.

Factors

What is the smallest number that has exactly 30 factors?

How many factors does 10 have?

Find some more numbers with 4 factors.

What is the same about numbers that have 4 factors?

What about numbers with other numbers of factors?

Foster, C. (2016). Thirty factors. *Mathematics in School, 45*(2),25–27.

What happens if you do

boring exercise,
boring exercise,
boring exercise,
boring exercise,
...

... it becomes interesting! (sometimes!)

A set of boring exercises?

How many factors does 10 have?
How many factors does 100 have?
How many factors does 1000 have?
How many factors does 10000 have?

Interesting exercises!

How many factors does 10 have?
How many factors does 100 have?
How many factors does 1000 have?
How many factors does 10000 have?

How many factors do the numbers in these sequences have? Why?

2, 4, 8, 16, 32, ...
5, 50, 500, 5000, ...
1, 4, 9, 16, 25, ...

What other sequences can you explore?

Integer Answers

What possible integers can go in the box so that this expression factorises?

$$x^2 + \square x - 12$$

How many solutions are there? Why?

What about in this case?

$$x^2 + 12x + \square$$

Factorise

$$x^2 + 7x + 6 = (x + 1)(x + 6)$$

$$2x^2 + 7x + 6 = (2x + 3)(x + 2)$$

Is this a fluke?
Can you make up another pair of quadratic expressions like this that both factorise?

What can you find out about how to make these?

Foster, C. (2012). Quadratic doublets. *Mathematical Gazette, 96*(536), 264–266.

Sums of Pairs

I am thinking of four numbers.

When I add every possible pair of numbers, I get the six answers:

2, 3, 4, 5, 6 and 7.

What might my four numbers be?

How many possible answers are there?

Foster, C. (2016). Sums of pairs. *Symmetry Plus, 59*, 14–16.

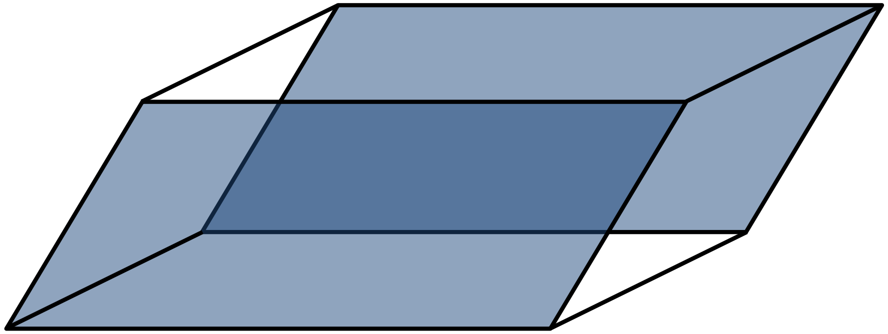
Nets

How many nets of a cube are there?

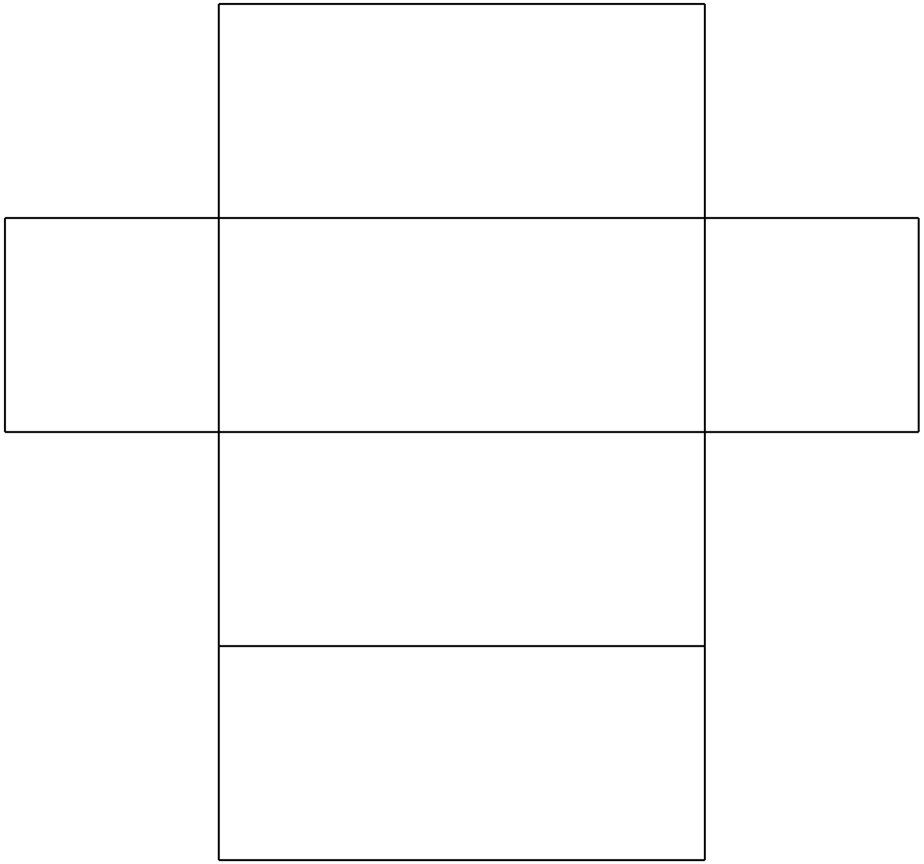
How do you make a net for a cuboid?

How do you make a net for a parallelepiped?

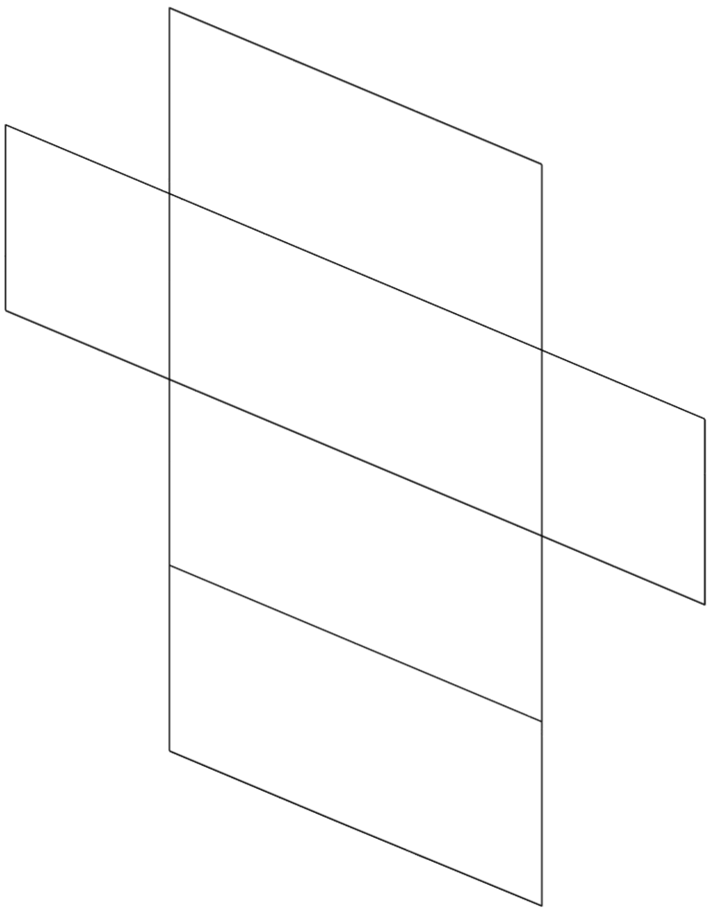
A what?



Nets



Nets



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