Problem Solving is more than Solving Problems

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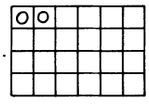
One of the fascinating outcomes, especially for teachers, of indulging in the business of problem solving activities with children is that the desire to reformulate and extend a given problem becomes irresistible. What follows is an example of this.

Towards a Solution

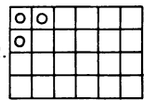
I was talking with some teachers about the problem solving work we had been doing. One of them explained an approach to the milk crate example (put 18 milk bottles into a 6×4 crate so that the number of bottles in all the rows and columns is always even) he had hit upon when he was trying to imagine how his children might have tackled it.

"I get some milk bottles and a crate.
I need to keep the rows and columns even.
I put in two bottles (Step 1)

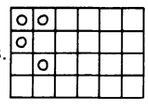
1. 

The rows are even

I must make the columns even
I make the first column even (Step 2)

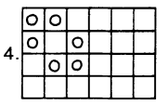
2. 

I make the second column even but I'll have a change (Step 3)

3. 

I have two even columns, one even row and two uneven rows

I must make the rows even (Step 4)

4. 

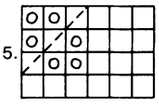
I have three even columns and three even rows

I could continue to put milk bottles into the crate, trying to preserve evenness as I go, but Ah Ha — I have put six bottles into the crate so I have 18 spaces so I have solved the problem because these spaces could be where the bottles really are."

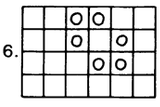
An Alternative View

In a sense the answer was arrived at by chance using the principle of keeping on putting in bottles to keep the number in each row and column even, but it also demonstrates the way in which a sudden insight can be arrived at while following a systematically chosen course of action. (I find that most adults tackling this problem start by filling in a complete row or column, and some interesting ideas can come out of this too.)

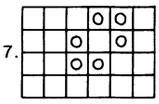
But this children's way made me ask some questions:
— Could the start have been made with the two bottles in any other positions than the two top left-hand ones?
— What would have happened had the change not been looked for in Step 3?
Before I had time to investigate these I noticed that there is a symmetry in the placing of the six bottles.

5. 

This led me to a third question which I investigated first:
— How important is this pattern?
Let's try some more possible positions using it:

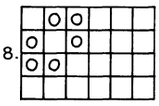
6. 

(a translation)

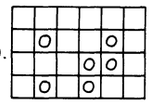
7. 

(a rotation and a translation)

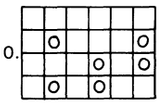
Some more questions bubble up:
— Is this the only pattern we can use?
— In how many ways can I place two sets of bottles in the crate so that the conditions of the problem are satisfied?
Here are some more solutions found by children:

8. 

(rotation)

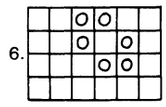
9. 

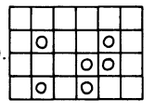
(rotation, translation, and enlargement of one set)
The symmetry has now disappeared

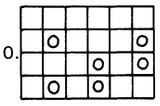
10. 

(and a one-way stretch as well as an enlargement)
Again the symmetry is lost.

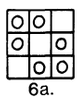
I showed these ideas to Harriet, who saw the solutions differently.
She said: I don't see your enlargements and one-way stretches. I see empty rows and columns.

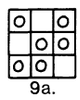
6. 

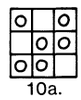
9. 

10. 

We stared at the grids.
Harriet said: If we get rid of the empty rows and columns we get a 3×3 grid:


6a.


9a.


10a.

Janet said: Three empty spaces in a 3×3 grid.
Harriet said: Those are bottles — three crucial bottles, which are the key to drawing the different arrangements.

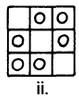
Harriet's analysis:

Having read Janet's extension of the problem to one of considering the patterns made by the six spaces, I wanted to investigate it further. With my class away for the holidays I have had to set about it myself! Here is my line of thought:

(1) The initial problem was to put 18 bottles into a 6×4 crate keeping the numbers in each row and column even.

(2) This was then reformulated to look at the 6 remaining spaces. These can be arranged with 2 spaces '○' in each of 3 rows and 3 columns as follows:

i.


ii.

iii.

iv.

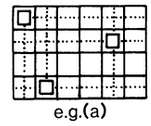
v.

vi.

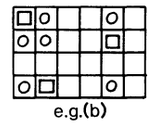
(i appears in Janet's 5; ii in her numbers 6, 7, 8 and iii in 9, 10 if you ignore blank columns.)

(3) In writing out the possible arrangements of the six ○'s shown in i-vi I was in fact thinking of the positions for the *3 gaps*. These gaps designate three milk bottles which can be seen as crucial in generating different arrangements.

e.g. Put 3 bottles '□' into the crate so that each is in a separate row and column.


e.g.(a)

Mark the six spaces '○' so that each is in the same row as one bottle '□', and the same column as another; i.e. where the dotted lines cross.


e.g.(b)

(In this particular example the squares so far filled make up pattern v when reduced to a 3×3 array.)
The squares left blank designate the remaining 15 bottles.

Using this method, we can succinctly calculate how many solutions there are since we only have to consider the first 3 bottles. The first bottle can go into any one of 24 places, but blocks the 9 places in its own row and column. The second bottle can go in any of the remaining 15 places but blocks a further 7. The third bottle must go in one of the 8 places left. The three bottles are indistinguishable and could have been put into those positions in any order. This gives the total number of solutions as

$$\frac{24 \times 15 \times 8}{6}$$

which equals 480. But the crate has rotational symmetry so the number of distinct solutions reduces to 240.

It is interesting to notice that the problem of

18 bottles in a 6×4 crate reduced to
6 spaces in a 6×4 grid reduced to
6 spaces in a 3×3 grid reduced to
3 bottles in a 3×3 grid and finally climbed back to
3 bottles in a 6×4 crate.

What was memorable to us in the exercise was
— our excitement following the insight which showed that 6 bottles could be replaced by 6 spaces to establish a solution,
— the anxiety caused by our two different approaches to the first solution,
— the satisfaction we felt from resolving this anxiety through discussion,
— the fascinating way in which this led us to an unexpected analysis of possible solutions.

Problem solving is more than solving problems.
It includes discussion and dispute, and forces us to recognise that different approaches are both possible and desirable. For us as teachers this brings a new dimension into both our teaching and our perception of children's work.

(The problem was taken from the Problem Library, produced after the research project *The Skills and Procedures of Mathematical Problem Solving* was completed. Copies can be obtained from Dr Leone Burton, Avery Hill College, Bexley Road, Eltham SE9 2PQ).