A conic theorem generalised: directed angles and applications.

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1 Introduction

A new and general proof of theorem [1] is presented, which accommodates directed angles and covers all possible configurations of line and conic. A sign convention for the use of directed angles in this setting is developed, and examples are given showing how this fundamental theorem can be used to establish, much more directly and succinctly, the many tangent/focus properties of conics.

In this article, angles whose directionality needs to be taken into account in equations, are indicated by a $\frown$ symbol, for example $\widehat{\varepsilon}$. Angles having a specified direction are indicated by a clockwise or anticlockwise symbol, for example $\overset{\curvearrowleft}{\alpha}$.

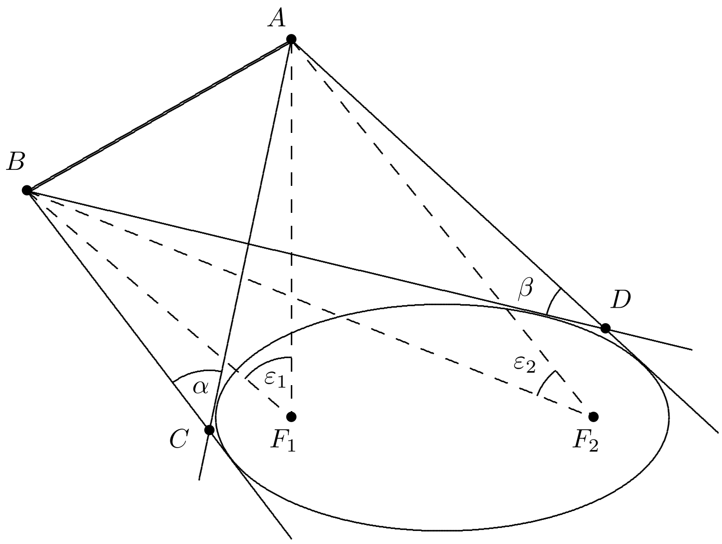

Figure 1: $\alpha + \beta = \varepsilon_1 + \varepsilon_2$

THEOREM: Let AC and AD be the tangents from A to a conic with foci F_1 and F_2. Let BC and BD be the tangents from B. Then $\angle\ A\widehat{C}B + \angle\ A\widehat{D}B = \angle\ A\widehat{F_1}B + \angle\ A\widehat{F_2}B$, where $\angle\ X\widehat{Y}Z$ is the rotation modulo 180° that carries line YX into line YZ.

PROOF: We need to prove $\widehat{\alpha} + \widehat{\beta} = \widehat{\varepsilon_1} + \widehat{\varepsilon_2}$ (see Figure 1).

Take axes with origin at the midpoint of F_1F_2, and x-axis in the direction $\overrightarrow{F_1F_2}$ (if $F_1 = F_2$ then any x-axis direction will do).

Let $A(x_a, y_a)$ and $B(x_b, y_b)$ be two points in the plane of, and outside, an ellipse with foci $F_1(x_{f_1}, y_{f_1})$ and $F_2(x_{f_2}, y_{f_2})$. Let AC, AD, BC, BD have gradients m_1, m_2, m_3, m_4 respectively. Let α be the angle between the tangents with gradients m_1 and m_3, and let β be the angle between the tangents with gradients m_2 and m_4. Let the segment AB subtend the angles ε_1 and ε_2 at the foci F_1 and F_2 respectively.

$$\begin{aligned}\alpha &= \tan^{-1}(m_3) - \tan^{-1}(m_1)\\ \beta &= \tan^{-1}(m_4) - \tan^{-1}(m_2)\end{aligned}$$

Let AB be allocated the arbitrary direction $\overrightarrow{AB}$. Since C, D, F_1, F_2 are all on the same side of $\overrightarrow{AB}$, they must all have the same angular direction and it therefore suffices, at this stage, to prove that $\alpha + \beta = \varepsilon_1 + \varepsilon_1$.

$$\alpha + \beta = \left\{\tan^{-1}(m_3) - \tan^{-1}(m_1)\right\} + \left\{\tan^{-1}(m_4) - \tan^{-1}(m_2)\right\}. \tag{1}$$

Rearranging gives

$$\begin{aligned}\alpha + \beta &= \left\{\tan^{-1}(m_3) + \tan^{-1}(m_4)\right\} - \left\{\tan^{-1}(m_1) + \tan^{-1}(m_2)\right\}\\ &= \tan^{-1}\left\{\frac{m_3 + m_4}{1 - m_3m_4}\right\} - \tan^{-1}\left\{\frac{m_1 + m_2}{1 - m_1m_2}\right\}. \end{aligned}\tag{2}$$

It is standard (see [2], p 248) that the gradients of the two tangents from $P(x_p, y_p)$ to the ellipse $x^2/a^2 + y^2/b^2 = 1$ are the roots of

$$m^2(x_p^2 - a^2) - m(2x_py_p) + y_p^2 - b^2 = 0. \tag{3}$$

Using (3) the sum and product of the gradients of tangents from A (m_1, m_2) and B (m_3, m_4) are therefore as follows.

$$m_1 + m_2 = \frac{2x_ay_a}{x_a^2 - a^2} \qquad\qquad m_1m_2 = \frac{y_a^2 - b^2}{x_a^2 - a^2},$$

$$m_3 + m_4 = \frac{2x_b y_b}{x_b^2 - a^2} \qquad\qquad m_3 m_4 = \frac{y_b^2 - b^2}{x_b^2 - a^2}.$$

Incorporating these into (2), and substituting the conic relationship $a^2 - b^2 = a^2e^2$ where e is the linear eccentricity, gives

$$\begin{aligned}
\alpha + \beta &= \tan^{-1}\left\{\frac{2x_b y_b}{x_b^2 - a^2e^2 - y_b^2}\right\} - \tan^{-1}\left\{\frac{2x_a y_a}{x_a^2 - a^2e^2 - y_a^2}\right\} \\
&= \tan^{-1}\left\{\frac{y_b(x_b - ae) + y_b(x_b + ae)}{(x_b - ae)(x_b + ae) - y_b^2}\right\} \\
&\qquad - \tan^{-1}\left\{\frac{y_a(x_a - ae) + y_a(x_a + ae)}{(x_a - ae)(x_a + ae) - y_a^2}\right\} \\
&= \tan^{-1}\left\{\frac{\left(\frac{y_b}{x_b+ae}\right) + \left(\frac{y_b}{x_b-ae}\right)}{1 - \left(\frac{y_b}{x_b+ae}\right)\left(\frac{y_b}{x_b-ae}\right)}\right\} \\
&\qquad - \tan^{-1}\left\{\frac{\left(\frac{y_a}{x_a+ae}\right) + \left(\frac{y_a}{x_a-ae}\right)}{1 - \left(\frac{y_a}{x_a+ae}\right)\left(\frac{y_a}{x_a-ae}\right)}\right\}.
\end{aligned} \tag{4}$$

Bearing in mind the coordinates of the foci $F_1 = (-ae, 0)$ and $F_2 = (+ae, 0)$, one can calculate the gradients of the lines from A and B to the foci. For example,

$$m_{af_2} = \frac{y_a - y_{f_2}}{x_a - x_{f_2}} = \frac{y_a}{x_a - ae}$$

where m_{af_2} is the gradient of the line AF_2. Equation 4 can therefore be expressed as follows.

$$\alpha + \beta = \tan^{-1}\left\{\frac{m_{bf_1} + m_{bf_2}}{1 - m_{bf_1}m_{bf_2}}\right\} - \tan^{-1}\left\{\frac{m_{af_1} + m_{af_2}}{1 - m_{af_1}m_{af_2}}\right\} \tag{5}$$

$$= \left\{\tan^{-1}(m_{bf_1}) + \tan^{-1}(m_{bf_2})\right\} - \left\{\tan^{-1}(m_{af_1}) + \tan^{-1}(m_{af_2})\right\}.$$

Rearranging gives

$$\alpha + \beta = \left\{\tan^{-1}(m_{bf_1}) - \tan^{-1}(m_{af_1})\right\} + \left\{\tan^{-1}(m_{bf_2}) - \tan^{-1}(m_{af_2})\right\}, \tag{6}$$

from which it follows

$$\alpha + \beta = \varepsilon_1 + \varepsilon_2 \tag{7}$$

as required.

Hyperbola

Since the hyperbola is obtained from the ellipse by replacing b^2 by $-b^2$, it follows that the above result also applies to the hyperbola.

Circle

In the case of a circle, both foci coincide at the centre, and so, if $\varepsilon_1 = \varepsilon_2 = \varepsilon$ then (7) becomes $\alpha + \beta = 2\varepsilon$.

Parabola

The parabola ($y^2 = 4ax$) is a special case; let its finite focus be $F_1(a, 0)$. In this case (see [2]; p 189) the gradients of the two tangents from a point $P(x_p, y_p)$ are the roots of

$$m^2 x_p - m y_p + a = 0. \tag{8}$$

Using (8) the sum and products of the gradients of tangents from $A(x_a, y_a)$ and $B(x_b, y_b)$ are therefore as follows.

$$m_1 + m_2 = \frac{y_a}{x_a} \qquad m_1 m_2 = \frac{a}{x_a},$$
$$m_3 + m_4 = \frac{y_b}{x_b} \qquad m_3 m_4 = \frac{a}{x_b}.$$

Substituting these values in (2) and rearranging gives

$$\begin{aligned} \alpha + \beta &= \tan^{-1}\left\{\frac{y_b}{x_b - a}\right\} - \tan^{-1}\left\{\frac{y_a}{x_a - a}\right\} \\ &= \tan^{-1}(m_{bf_1}) - \tan^{-1}(m_{af_1}), \end{aligned}$$

from which it follows

$$\alpha + \beta = \varepsilon_1.$$

Comparing this result with (6) and (7), it follows that in the parabola case

$$\varepsilon_2 = 0 = \tan^{-1}(m_{bf_2}) - \tan^{-1}(m_{af_2}),$$

and so $m_{af_2} = m_{bf_2}$ and hence lines AF_2 and BF_2 are parallel, which is consistent with the notion that the parabola's other focus F_2 is at infinity.

2 Directed angles and line segments

The power of directed line segments to unify certain areas of geometry has been well documented by Klein [3]. Similarly, it is useful in this context to regard certain angles as also being directed, and to reference their direction in relation to that of a directed line segment subtending the angles.

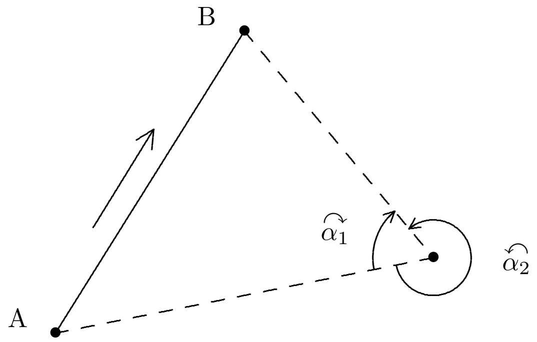

Figure 2: Internal and external directed angles in relation to the line $\overrightarrow{AB}$.

For the purposes of this article we shall adopt the rule that the direction of the internal angle subtended by the directed line segment $\overrightarrow{AB}$ is in the same sense as the directed line segment. For example, in Figure 2 the angle $\overset{\curvearrowright}{\alpha_1}$ is therefore considered to be clockwise and $\overset{\curvearrowleft}{\alpha_2}$ is anticlockwise, and since they can both be viewed as having the same net effect it follows that $\overset{\curvearrowright}{\alpha_1} = \overset{\curvearrowleft}{\alpha_2}$. Similarly, since the combination of $\overset{\curvearrowright}{\alpha_1}$ followed by $\overset{\curvearrowright}{\alpha_2}$ *in the same direction* returns us to the initial position, it follows that $\overset{\curvearrowright}{\alpha_1} + \overset{\curvearrowright}{\alpha_2} = \overset{\curvearrowright}{\alpha_1} - \overset{\curvearrowleft}{\alpha_2} = 0$.

Using this idea, Figure 3 shows how we can generalise the rule to include the alternative pairing of tangents from A and B to the conic. Simple geometry shows that in this particular case

$$\alpha + \beta = \gamma - \delta. \tag{9}$$

Note that this can also be derived by simply rearranging the right-hand side of (1) as follows.

$$\alpha + \beta \; = \; \left\{ \tan^{-1}(m_2) - \tan^{-1}(m_3) \right\} - \left\{ \tan^{-1}(m_4) - \tan^{-1}(m_1) \right\} \; = \; \gamma - \delta.$$

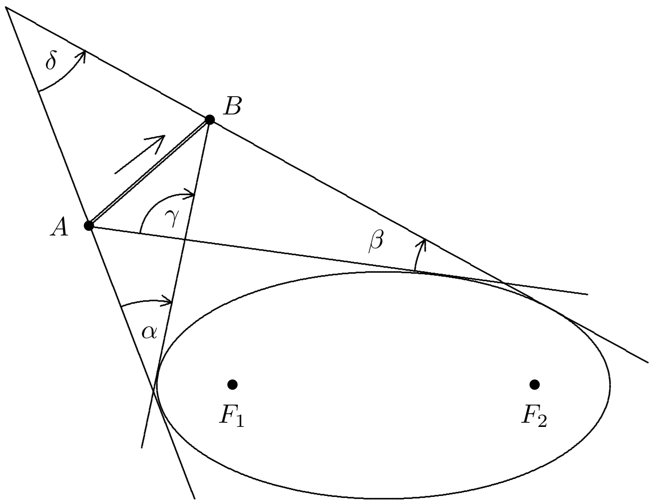

Figure 3:

Significantly, not only do γ and δ have opposite signs in (9), but they are also on opposite sides of the line AB, and have opposite angular directions in relation to that of the directed line $\overrightarrow{AB}$ (see Figure 3). If we also adopt the rule that a true relationship between the absolute values of the angles is associated with a consistent angular direction (say, clockwise), then (9) can be expressed as follows.

$$\overset{\curvearrowright}{\alpha} + \overset{\curvearrowright}{\beta} = \overset{\curvearrowright}{\gamma} - \overset{\curvearrowright}{\delta}$$

Since $\overset{\curvearrowleft}{\delta} = -\overset{\curvearrowright}{\delta}$, then this is equivalent to

$$\overset{\curvearrowright}{\alpha} + \overset{\curvearrowright}{\beta} = \overset{\curvearrowright}{\gamma} + \overset{\curvearrowleft}{\delta}\ .$$

Thus by viewing the angles as directed, and referencing angular direction relative to that of the directed line $\overrightarrow{AB}$, we are able to generalise and say that the sums of each pair of *directed* angles are equal, as expressed in the following rule embracing all six angles.

$$\overset{\frown}{\alpha} + \overset{\frown}{\beta} = \overset{\frown}{\gamma} + \overset{\frown}{\delta} = \overset{\frown}{\varepsilon_1} + \overset{\frown}{\varepsilon_1}$$

3 Sign convention

We have therefore established a simple sign convention which allows the relationship between the absolute values of the various angles to be easily determined in particular cases, as follows:

- Firstly, allocate an *arbitrary* direction to an appropriate line (say, $\overrightarrow{AB}$).
- Secondly, give the subtended angles a direction in the same sense as the directed line, and apply these directions to the rule

$$\overset{\frown}{\alpha} + \overset{\frown}{\beta} = \overset{\frown}{\gamma} + \overset{\frown}{\delta} = \overset{\frown}{\varepsilon_1} + \overset{\frown}{\varepsilon_1}.$$

- Thirdly, adjust the equation so that there is a consistent angular direction. The angular directions can then be removed, leaving the correct relationship for the absolute values of the angles.

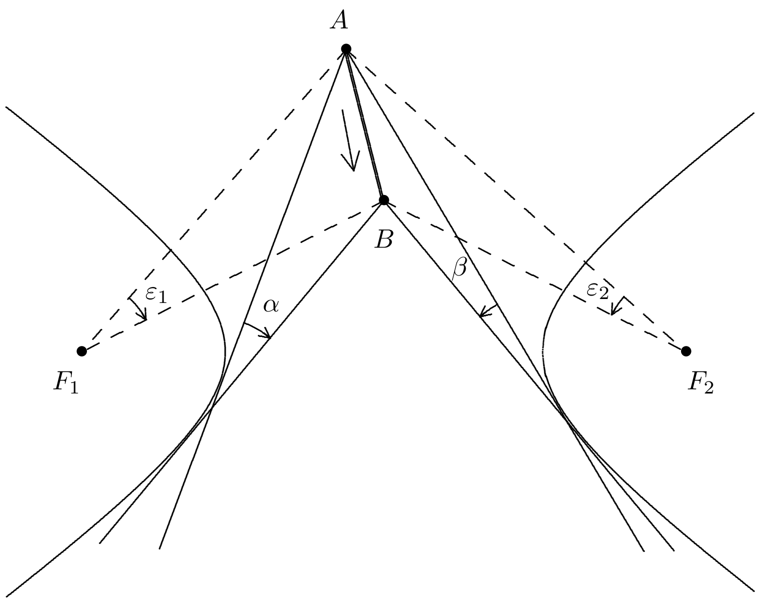

Figure 4:

For example, in the hyperbola case shown in Figure 4 the angles $\overset{\curvearrowright}{\varepsilon_1}$ and $\overset{\curvearrowright}{\alpha}$ are regarded as being clockwise, and the angles $\overset{\curvearrowleft}{\varepsilon_2}$ and $\overset{\curvearrowleft}{\beta}$ are anticlockwise. In this case applying these angular directions to the rule $\overset{\frown}{\alpha} + \overset{\frown}{\beta} = \overset{\frown}{\varepsilon_1} + \overset{\frown}{\varepsilon_2}$ gives

$$\overset{\curvearrowright}{\alpha} + \overset{\curvearrowleft}{\beta} = \overset{\curvearrowright}{\varepsilon_1} + \overset{\curvearrowleft}{\varepsilon_2}\ .$$

Rearranging to force a consistent angular direction (say, clockwise) gives

$$\overset{\curvearrowright}{\alpha} - \overset{\curvearrowright}{\beta} = \overset{\curvearrowright}{\varepsilon_1} - \overset{\curvearrowright}{\varepsilon_2},$$

which now yields the correct relationship for this particular configuration as follows

$$\alpha - \beta = \varepsilon_1 - \varepsilon_2.$$

In practice, although the subtended angles associated with the foci (ε_1, ε_2) are clearly defined, this is often not the case with regard to the angles associated with the tangents, particularly with unusual configurations of line and conic. For example, if one end of the line is on the curve, or the line passes through a focus, or is a tangent. In these situations, it is often helpful to observe how the various angles and their directions change as the directed line is moved from well outside the conic (as in Figures 1 or 4) to its final position.

Figure 5 illustrates an interesting example where both ends of the directed line $\overrightarrow{AB}$ lie on the curve. In this case, applying the theorem to the ellipse gives

$$\overset{\curvearrowright}{\alpha} + \overset{\curvearrowright}{\beta} = \overset{\curvearrowleft}{\varepsilon_1} + \overset{\curvearrowright}{\varepsilon_2}\ .$$

Rearranging to force a consistent clockwise orientation gives

$$\overset{\curvearrowright}{\alpha} + \overset{\curvearrowright}{\beta} = \overset{\curvearrowright}{(360 - \varepsilon_1)} + \overset{\curvearrowright}{\varepsilon_2},$$

$$\alpha + \beta \ = \ 360 - \varepsilon_1 + \varepsilon_2.$$

It follows that if the tangents are parallel (i.e. $\alpha = \beta = 180$), then $\varepsilon_1 = \varepsilon_2$.

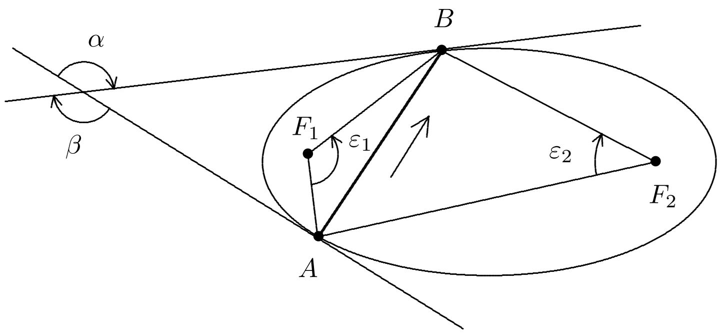

Figure 5:

4 Examples

Since this theorem highlights a fundamental conic relationship between lines, tangents and foci, it can be used to prove the various tangent/focus properties more efficiently and succinctly than the standard textbook methods, as shown in the following examples (see [1] also).

Example 1

The line joining the points of contact of parallel tangents to a circle is a diameter.

PROOF (see Figure 6).

Applying the theorem with respect to the line $\overrightarrow{AB}$ gives $\overset{\frown}{\alpha} + \overset{\frown}{\beta} = 2\,\overset{\frown}{\varepsilon}$.
Since $\alpha \equiv \beta \equiv 0 \pmod{180^\circ}$, then $2\varepsilon \equiv 0 \pmod{180^\circ}$, so $\varepsilon \equiv 0, \pm 90^\circ, \pm 180^\circ \pmod{360^\circ}$. But 0 is excluded ($A \neq B$), as is $\pm 90^\circ$ (the tangents are not mutually perpendicular), so $\varepsilon = 180^\circ$, i.e. AOB is a diameter.

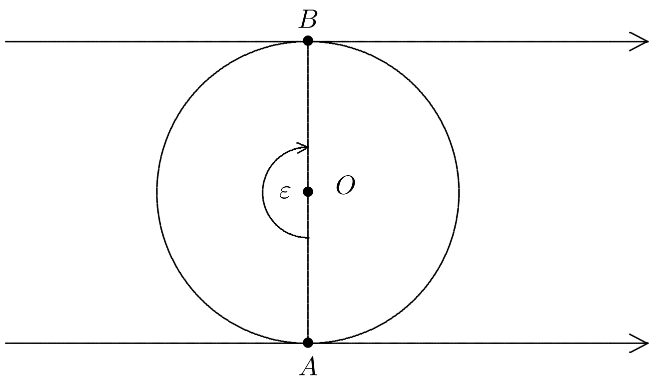

Figure 6:

Example 2

> *The circumcircle of the triangle formed by any three tangents to a parabola passes through the focus.*

PROOF (see Figure 7).

Applying the theorem with respect to the tangent $\overrightarrow{AB}$ gives $\overset{\curvearrowleft}{\alpha} + 0 = \overset{\curvearrowleft}{\varepsilon} + 0$,
so $\alpha = \varepsilon$. Since both angles are subtended by AB, it follows that A, B, F, C lie on a circle.

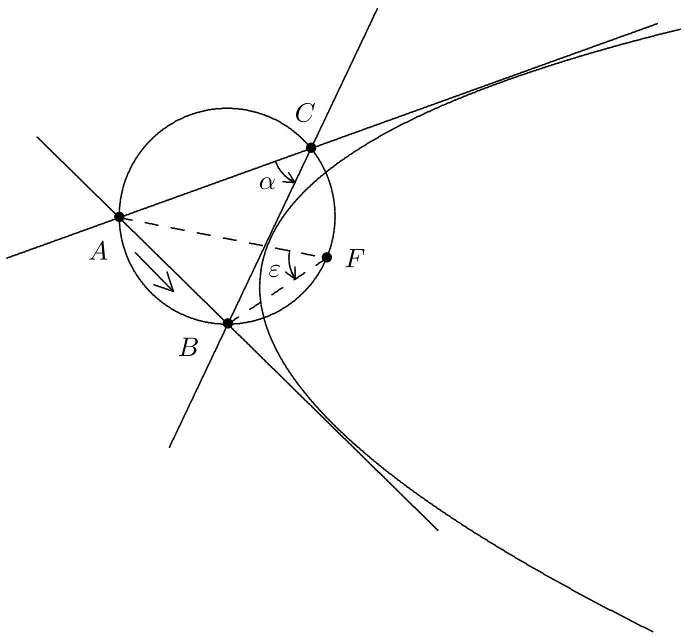

Figure 7:

Example 3

The locus of the foot of the perpendicular from the focus of a parabola to a tangent is the tangent at the apex.

PROOF (see Figure 8).

Let t be the tangent. Let B be the foot of the perpendicular to t from the focus F. Let the other tangent from B touch the parabola at A.

Applying the theorem with respect to the segment $\overrightarrow{AB}$ gives $\overset{\curvearrowright}{\alpha} + 0 = \overset{\curvearrowright}{\varepsilon} + 0$. Thus $\alpha = \varepsilon$. Since $\alpha + \psi = 90$, it follows that $\varepsilon + \psi = 90$. Angle FAB is therefore a right angle, and hence AB is the tangent at the apex of the parabola.

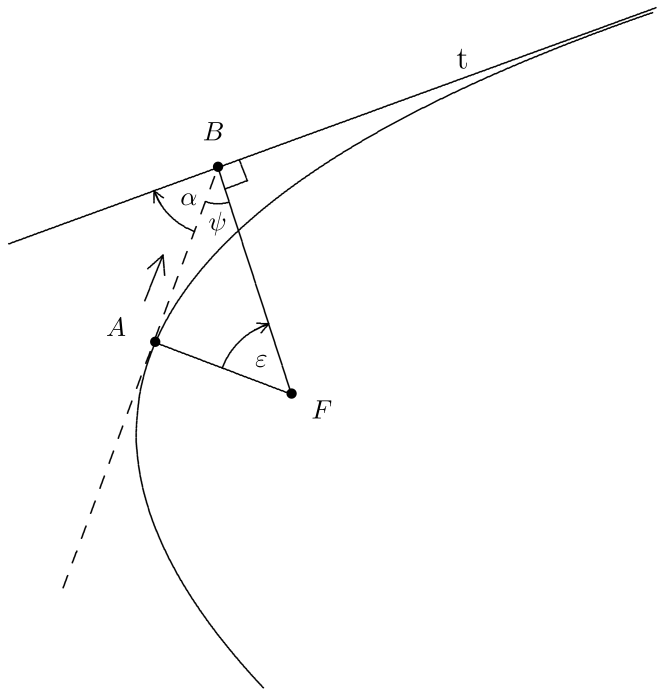

Figure 8:

5 Conclusion

This theorem crystalised from some experimental observations [4, 5] in the area of vision physiology—a rich source of interesting mathematical problems. Hopefully, this article gives a flavour of the power of directed angles and brings to life a curious theorem, which can be used to establish tangent-focus properties of conics via an unusual perspective.

6 Acknowledgements

Finally, I would like to thank the editor Steve Abbott and the anonymous referees and proof readers for their help and advice during the preparation of this article.

7 References

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